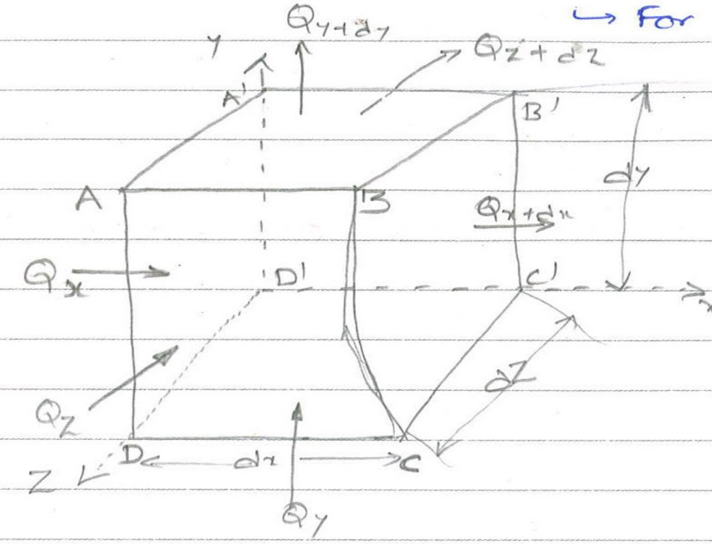
Heat Conduction through Cartesian, Cylindrical and Spherical Co-ordinates

- General Heat conduction equation:

(1) Cartesian Co-ordinates:

↳ For solids with rectangular bodies like Slabs, cubes etc.



Consider the flow of heat through an infinitesimal volume element oriented on 3D co-ordinate system.

The sides dx, dy, dz have been taken parallel to the x, y, z axis.

The energy balance for the little element as per 1st law of TD

Net heat conducted into element dxdydz per unit time (1) + Internal heat generated per unit time (2) = Increase in internal energy per unit time (3) + workdone by element per unit time (4) — (1)

The quantity of heat flowing through AA'DD' face during time interval $d\tau$ is

$$\text{Heat influx } Q_x = -k_x \cdot (dy \cdot dz) \cdot \frac{\partial t}{\partial x} \cdot d\tau$$

$$\text{Heat outflux } Q_{x+dx} = Q_x + \frac{\partial}{\partial x}(Q_x) \cdot dx .$$

Heat accumulated in x-direction

$$dQ_x = Q_x - Q_{x+dx}$$
$$= Q_x - \left(Q_x + \frac{\partial}{\partial x} \cdot Q_x \cdot dx\right)$$
$$= -\frac{\partial}{\partial x}\left[-K_x \cdot (dy \cdot dz) \cdot \frac{\partial t}{\partial x} \cdot d\tau\right] dx$$
$$= \frac{\partial}{\partial x} \cdot \left[K_x \cdot \frac{\partial t}{\partial x}\right] dx \cdot dy \cdot dz \cdot d\tau \quad \text{——} (2)$$

|||ly $$dQ_y = \frac{\partial}{\partial y} \cdot \left[K_y \cdot \frac{\partial t}{\partial y}\right] dx \cdot dy \cdot dz \cdot d\tau \quad \text{——} (3)$$

|||ly $$dQ_z = \frac{\partial}{\partial z}\left[K_z \cdot \frac{\partial t}{\partial z}\right] \cdot dx \cdot dy \cdot dz \cdot d\tau \quad \text{——} (4)$$

Total heat stored in the elemental volume due to heat flow along all axes is

(2) + (3) + (4)

$$\left[\frac{\partial}{\partial x}\left[K_x \cdot \frac{\partial t}{\partial x}\right] + \frac{\partial}{\partial y} \cdot \left[K_y \cdot \frac{\partial t}{\partial y}\right] + \frac{\partial}{\partial z}\left[K_z \cdot \frac{\partial t}{\partial z}\right]\right] dx \cdot dy \cdot dz \cdot d\tau \quad (5)$$

If q_g is heat generated per unit volume and per unit time then total heat generated in control volume $= q_g \cdot dx \cdot dy \cdot dz \cdot d\tau$ (6)

Heat flow along Co-ordinate axes and heat generated in control volume increases thermal/internal energy of the lattice.

Increase in thermal energy $= \underbrace{\rho \cdot (dx \cdot dy \cdot dz)}_{m} \cdot \underbrace{C}_{C_p} \cdot \underbrace{\frac{\partial t}{\partial \tau}}_{\Delta T} \cdot d\tau$ —— (7)

From energy balance Considerations: (5) + (6) = (7)

$$\left[\frac{\partial}{\partial x}\left(K_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \cdot \frac{\partial t}{\partial z}\right)\right] dx \cdot dy \cdot dz \cdot d\tau + q_g \cdot dx \cdot dy \cdot dz \cdot d\tau$$
$$= \rho \cdot dx \cdot dy \cdot dz \cdot C \cdot \frac{\partial t}{\partial \tau} \cdot d\tau$$

$$\frac{\partial}{\partial x} \cdot \left[K_x \cdot \frac{\partial t}{\partial x}\right] + \frac{\partial}{\partial y} \cdot \left[K_y \cdot \frac{\partial t}{\partial y}\right] + \frac{\partial}{\partial z} \cdot \left[K_z \cdot \frac{\partial t}{\partial z}\right] + q_g = \rho \cdot C \cdot \frac{\partial t}{\partial \tau}$$

↳ General heat conduction equation, establishes in differential form the relationship between time & space variation of temperatu at any point of the solid through which conduction takes place

↗ ρ, c, k are same every where in the material

- For homogeneous & isotropic material:

↳ These properties are not directional characteristics

Hence $k_x = k_y = k_z = k$ then

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{\rho c}{k} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

→ $\alpha = k/\rho c$ is called Thermal diffusivity

↳ Tells how fast heat is propagated during changes of temp. with

α higher ⇒ Time required for heat to penetrate is shorter

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \Rightarrow \text{Laplacian equation}$$

- Special forms of heat conduction equation:

(i) Steady state:

There is no dependency of temp. on time.

Heat flow equation $\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = 0$

$$\nabla^2 t + \frac{q_g}{k} = 0 \quad \text{(Poisson's equation)}$$

(ii) In absence of internal heat generation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

$$\nabla^2 t = 0 \quad \text{(Laplace equation)}$$

(iii) Unsteady state with no internal heat generation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

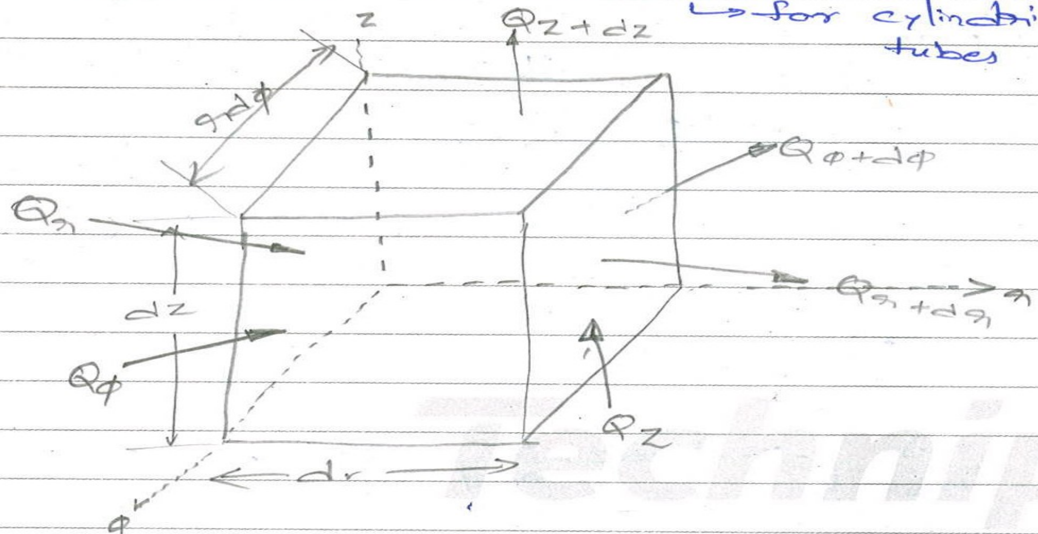
$$\nabla^2 t = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \text{(Fourier equation)}$$

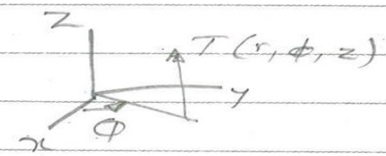
iii) For one-D & steady state heat flow with no internal heat generation, then

$$\cancel{\frac{d^2t}{dx^2} = Q} \qquad \frac{d^2t}{dx^2} = 0$$

→ Cylindrical Co-ordinates :-

↳ for cylindrical bodies like cylinders, tubes





Consider a small volume element having sides dr, dz & $rd\phi$ as shown in above fig.

Assuming material to be isotropic.

(i) radial direction $(r-\phi)$ plane :-

Heat influx $\quad Q_r = -k.(rd\phi.dz).\frac{\partial t}{\partial r}.d\tau$

Heat efflux $\quad Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\,dr$

Heat stored in the element due to flow of heat in radial direction

$$dQ_r = Q_r - Q_{r+dr}$$

$$= \frac{-\partial}{\partial r}(Q_r)\,dr$$

$$= \frac{-\partial}{\partial r}\left(-k.(rd\phi.dz).\frac{\partial t}{\partial r}.d\tau\right).dr$$

$$= k.(dr.d\phi.dz)\frac{\partial}{\partial r}\left(r.\frac{\partial t}{\partial r}\right)d\tau$$

$$= k.dr.d\phi.dz.\left[r.\frac{\partial^2 t}{\partial r^2}+\frac{\partial t}{\partial r}\right]d\tau$$

$$= k.(dr.rd\phi.dz)\left[\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}.\frac{\partial t}{\partial r}\right]d\tau$$

(ii) Tangential direction (r-z plane)

Heat influx $Q_\phi = -k.(dr.dz).\frac{\partial t}{rd\phi}.d\tau$

Heat efflux $Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r.\partial\phi}(Q_\phi).rd\phi$

Heat stored in tangential direction

$$dQ_\phi = Q_\phi - Q_{\phi+d\phi}$$

$$= \frac{-\partial}{r.\partial\phi}(Q_\phi).rd\phi$$

$$= \frac{-\partial}{r\partial\phi}\left[-k.dr.dz.\frac{\partial t}{r\partial\phi}.d\tau\right].rd\phi$$

$$= k.dr.d\phi.dz.\frac{\partial}{\partial\phi}\left[\frac{1}{r}.\frac{\partial t}{\partial\phi}\right].d\tau$$

$$= k.(dr.rd\phi.dz).\frac{1}{r^2}.\frac{\partial^2 t}{\partial\phi^2}d\tau$$

(iii) Axial direction (r-ϕ plane):

Heat influx $Q_z = -k.(rd\phi.dr).\frac{\partial t}{\partial z}.d\tau$

" efflux $Q_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q_z)dz$

Heat stored in the element due to heat flow in axial direction

$$dQ_z = Q_z - Q_{z+dz}$$

$$= -\frac{\partial}{\partial z}\left[-k.(rd\phi.dr).\frac{\partial t}{\partial z}.d\tau\right]dz$$

$$= k.(dr.rd\phi.dz)\frac{\partial^2 t}{\partial z^2}.d\tau$$

Heat generated with in control volume $= q_g \cdot dv \cdot d\tau$

Rate of change of energy with in control volume $= \rho \cdot dv \cdot c \cdot \frac{\partial t}{\partial \tau} \cdot d\tau$

from energy equation

$$K \cdot dv \cdot \left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right] d\tau + q_g \cdot dv \cdot d\tau$$

$$= \rho \cdot dv \cdot C \cdot \frac{\partial t}{\partial \tau} \cdot d\tau$$

Divide by $dv \cdot d\tau$ on both sides

$$K \cdot \left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right] + q_g = \rho \cdot C \cdot \frac{\partial t}{\partial \tau}$$

$$\boxed{\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{q_g}{K} = \frac{\rho C}{K} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}}$$

The heat conduction in cylindrical co-ordinates could also be obtained by doing following co-ordinate transformation

$$x = r\cos\phi \quad ; \quad y = r\sin\phi \quad ; \quad z = z$$

using chain rule $\quad \frac{\partial t}{\partial r} = \frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial r}$

$$= \frac{\partial t}{\partial x} \cdot \cos\phi + \frac{\partial t}{\partial y} \cdot \sin\phi$$

$$\cos\phi \cdot \frac{\partial t}{\partial r} = \cos^2\phi \cdot \frac{\partial t}{\partial x} + \sin\phi \cdot \cos\phi \cdot \frac{\partial t}{\partial y} \quad \text{--- (1)}$$

Also $\quad \frac{\partial t}{\partial \phi} = \frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial \phi}$

$$= \frac{\partial t}{\partial x} \cdot (-r\sin\phi) + \frac{\partial t}{\partial y} \cdot (r\cos\phi)$$

$$\frac{\sin\phi}{r} \cdot \frac{\partial t}{\partial \phi} = -\sin^2\phi \cdot \frac{\partial t}{\partial x} + \sin\phi \cdot \cos\phi \cdot \frac{\partial t}{\partial y} \quad \text{--- (2)}$$

from ① & ②

$$\frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi} = -\sin^2\phi\,\frac{\partial t}{\partial x} + \left[\cos\phi\cdot\frac{\partial t}{\partial r} - \cos^2\phi\cdot\frac{\partial t}{\partial x}\right]$$

$$= -\frac{\partial t}{\partial x} + \cos\phi\cdot\frac{\partial t}{\partial r}$$

$$\frac{\partial t}{\partial x} = \cos\phi\cdot\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi} \quad — ③$$

upon diff. with respect to x.

$$\frac{\partial}{\partial x}\cdot\left(\frac{\partial t}{\partial x}\right) = \frac{\partial}{\partial x}\left[\cos\phi\cdot\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi}\right]$$

$$\frac{\partial^2 t}{\partial x^2} = \cos\phi\cdot\frac{\partial}{\partial r}\left[\frac{\partial t}{\partial x}\right] - \frac{\sin\phi}{r}\cdot\frac{\partial}{\partial\phi}\cdot\left[\frac{\partial t}{\partial x}\right].$$

$$= \cos\phi\cdot\frac{\partial}{\partial r}\left[\cos\phi\cdot\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi}\right] - \frac{\sin\phi}{r}\cdot\frac{\partial}{\partial\phi}\left[\cos\phi\,\frac{\partial t}{\partial r} - \frac{\sin\phi}{r}\cdot\frac{\partial t}{\partial\phi}\right]$$

$\left[\frac{\partial}{\partial r}\cdot\frac{1}{r} = \frac{-1}{r^2}\right]$

$\left[\frac{\partial}{\partial\phi}\cdot\cos\phi = \sin\phi\right]$

$$= \cos^2\phi\cdot\frac{\partial^2 t}{\partial r^2} + \frac{\cos\phi\cdot\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} + \frac{\sin^2\phi}{r}\,\frac{\partial t}{\partial\phi} + \frac{\sin^2\phi}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2} + \frac{\sin\phi\cdot\cos\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} \quad — ④$$

Illy

$$\frac{\partial^2 t}{\partial y^2} = \sin^2\phi\cdot\frac{\partial^2 t}{\partial r^2} + \frac{\cos^2\phi}{r}\cdot\frac{\partial t}{\partial r} - \frac{\cos\phi\cdot\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} + \frac{\cos^2\phi}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2} - \frac{\cos\phi\cdot\sin\phi}{r^2}\cdot\frac{\partial t}{\partial\phi} \quad — ⑤$$

from ④ & ⑤

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial\phi^2} \quad — ⑥$$

Substituting ⑥ in $\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{K} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial\tau}$

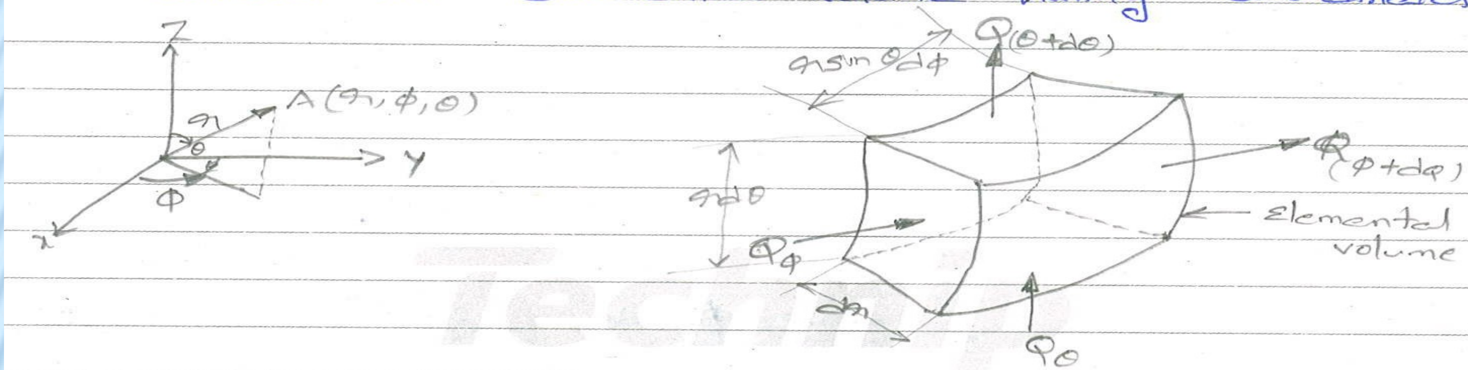
$$\left[\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}\right] + \frac{q_g}{K} = \frac{1}{\alpha}\cdot\frac{\partial t}{\partial \tau}$$

– For steady state uni-direction heat flow in the radial direction, and with no internal heat generation

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} = 0$$

→ Spherical Co-ordinates:-

Consider an elemental volume having Co-ordinates (r, ϕ, θ):



volume of the element $= dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$

q_g = heat generated per unit volume per unit time.

i) Net heat accumulated in the element:–

Heat flow through $r-\theta$ plane: ϕ-direction:

Heat influx, $Q_\phi = -K\cdot(dr \cdot r\,d\theta)\cdot\frac{\partial t}{r\sin\theta\cdot\partial\phi}\cdot d\tau$

Heat efflux, $Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\sin\theta\,\partial\phi}(Q_\phi)\cdot r\sin\theta\cdot d\phi$

Heat accumulated in ϕ-direction,

$$dQ_\phi = Q_\phi - Q_{\phi+d\phi}$$

$$= \frac{-1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\cdot(Q_\phi)\cdot r\sin\theta\cdot d\phi$$

$$= \frac{-1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left[-K\cdot(dr\cdot r\,d\theta)\frac{1}{r\sin\theta}\cdot\frac{\partial t}{\partial\phi}\cdot d\tau\right)\cdot r\sin\theta\cdot d\phi$$

$$= K\cdot[dr\cdot r\,d\theta\cdot r\sin\theta\cdot d\phi]\,\frac{1}{r^2\cdot\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2}\cdot d\tau$$

Heat flow in r-ϕ plane, θ-direction:

Heat influx, $Q_\theta = -K\cdot(dr\cdot r\sin\theta\,d\phi)\cdot\frac{\partial t}{r\,\partial\theta}\cdot d\tau$

Heat efflux, $Q_{\theta+d\theta} = Q_\theta' + \frac{\partial}{r\,\partial\theta}(Q_\theta'')\,r\,d\theta$

Heat accumulated in θ-direction:

$$dQ_\theta^* = Q_\theta - Q_{\theta+d\theta}$$

$$= \frac{-\partial}{r\,\partial\theta}(Q_\theta)\,r\,d\theta$$

$$= \frac{-\partial}{r\,\partial\theta}\left[-K\cdot(dr\cdot r\sin\theta\cdot d\phi)\frac{\partial t}{r\,\partial\theta}\cdot d\tau\right]r\cdot d\theta$$

$$= \frac{K}{r}\cdot\frac{dr\cdot r\,d\phi\cdot r\,d\theta}{r}\cdot\frac{\partial}{\partial\theta}\cdot\left[\sin\theta\cdot\frac{\partial t}{\partial\theta}\right]d\tau$$

$$= K\cdot dr\cdot r\,d\theta\cdot r\sin\theta\cdot d\phi\cdot\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta\cdot\frac{\partial t}{\partial\theta}\right]d\tau$$

Heat flow in θ-ϕ plane, r-direction:

Heat influx $Q_r^* = -K\cdot(r\,d\theta\cdot r\sin\theta\,d\phi)\cdot\frac{\partial t}{\partial r}\cdot d\tau$

" efflux $Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\,dr$

Heat accumulated in r-direction,

$$dQ_r = Q_r - Q_{(r+dr)}$$

$$= \frac{-\partial}{\partial r}[Q_r]\,dr$$

$$= \frac{-\partial}{\partial r}\left[-K\cdot(r\,d\theta\cdot r\sin\theta\cdot d\phi)\frac{\partial t}{\partial r}\cdot d\tau\right]dr$$

$$= K\cdot d\theta\cdot\sin\theta\cdot d\phi\cdot dr\cdot\frac{\partial}{\partial r}\left[r^2\cdot\frac{\partial t}{\partial r}\right]\cdot d\tau$$

Net heat accumulated in the element

$$= k \cdot dr \cdot r d\theta \cdot r\sin\theta \, d\phi \left\{ \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\sin\theta \cdot \frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left[r^2 \cdot \frac{\partial t}{\partial r}\right]\right] d\tau$$

(ii) Heat generated with in the element

$$= q_g \cdot (dr \cdot r d\theta \cdot r\sin\theta \cdot d\phi \cdot) \, d\tau$$

(iii) Energy stored in the element

The increase in thermal energy $= \rho \cdot dr \cdot r d\theta \cdot r\sin\theta d\phi \cdot c \cdot \frac{\partial t}{\partial \tau} d\tau$

from energy balance equation (1), (2), (3) ;

(1) + (2) = (3) ; DOBS by $k \cdot [dr \cdot r d\theta \cdot r\sin\theta \cdot d\phi] d\tau$,

$$\boxed{\left[\frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left[\sin\theta \cdot \frac{\partial t}{\partial \theta}\right] + \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left[r^2 \frac{\partial t}{\partial r}\right]\right] + \frac{q_g}{k} = \frac{\rho c}{k} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}}$$

- For steady, one-D, no heat sources

$$\frac{1}{r^2} \cdot \frac{d}{dr}\left[r^2 \cdot \frac{dt}{dr}\right) = 0$$

The above equation can also be obtained by using

$x = r\sin\theta \cdot \sin\phi$, $y = r\sin\theta \cdot \cos\phi$, $z = r\cos\theta$.