Transient Conduction: The Lumped Capacitance Method

Chapter Five

Sections 5.1 thru 5.3

Transient Conduction

- A heat transfer process for which the temperature varies with time, as well as location within a solid.

- It is initiated whenever a system experiences a change in operating conditions and proceeds until a new steady state (thermal equilibrium) is achieved.

- It can be induced by changes in:
  - surface convection conditions (h, T_∞),
  - surface radiation conditions (h_r, T_{sur}),
  - a surface temperature or heat flux, and/or
  - internal energy generation.

- Solution Techniques
  - The Lumped Capacitance Method
  - Exact Solutions
  - The Finite-Difference Method

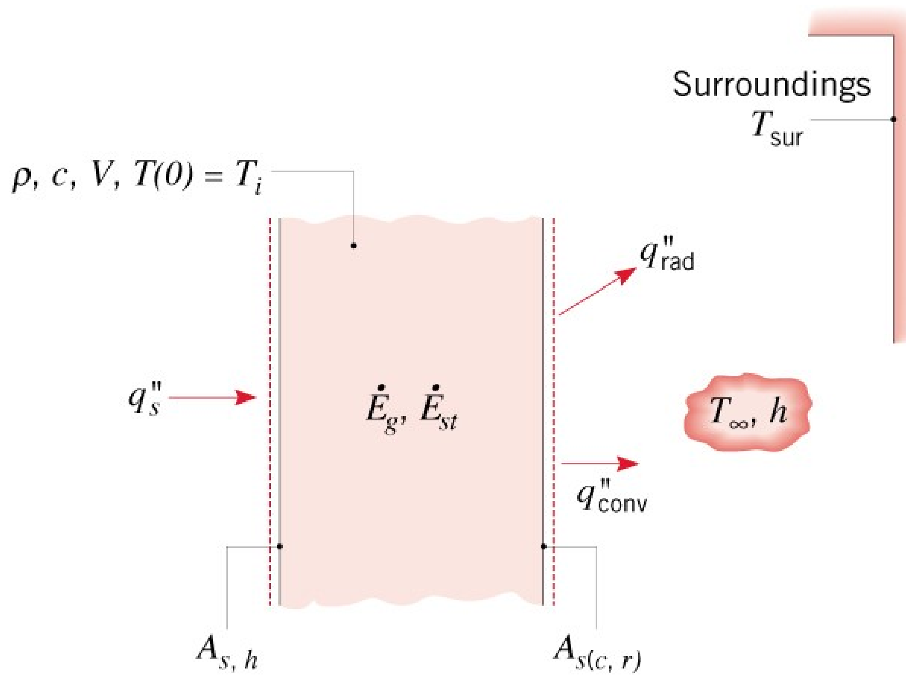
The Lumped Capacitance Method

- Based on the assumption of a spatially uniform temperature distribution throughout the transient process. Hence $T(\vec{r},t) \approx T(t)$.

- Why is the assumption never fully realized in practice?

- General Lumped Capacitance Analysis:

  - Consider a general case, which includes convection, radiation and/or an applied heat flux at specified surfaces $(A_{s,c}, A_{s,r}, A_{s,h})$, as well as internal energy generation



- First Law:

$$\frac{dE_{st}}{dt} = \rho \forall c \frac{dT}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g$$

- Assuming energy outflow due to convection and radiation and with inflow due to an applied heat flux q_s'',

$$\rho \forall c \frac{dT}{dt} = q_s'' A_{s,h} - h A_{s,c} \left(T - T_\infty \right) - h_r A_{s,r} \left(T - T_{sur} \right) + \dot{E}_g$$

- Is this expression applicable in situations for which convection and/or radiation provide for energy inflow?

- May h and h_r be assumed to be constant throughout the transient process?

- How must such an equation be solved?

- Special Cases (Exact Solutions, $T(0) \equiv T_i$)
  - Negligible Radiation $(\theta \equiv T - T_\infty,\ \theta' \equiv \theta - b/a)$:

$$a \equiv hA_{s,c} / \rho \forall c \qquad\qquad b \equiv \left(q_s'' A_{s,h} + \dot{E}_g \right) / \rho \forall c$$

The non-homogeneous differential equation is transformed into a homogeneous equation of the form:

$$\frac{d\theta'}{dt} = -a\theta'$$

Integrating from *t=0 to any* t and rearranging,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty}\left[1 - \exp(-at)\right] \tag{5.25}$$

To what does the foregoing equation reduce as steady state is approached?

How else may the steady-state solution be obtained?

- Negligible Radiation and Source Terms $\left(h >> h_r,\ \dot{E}_g = 0,\ q_s'' = 0\right)$:

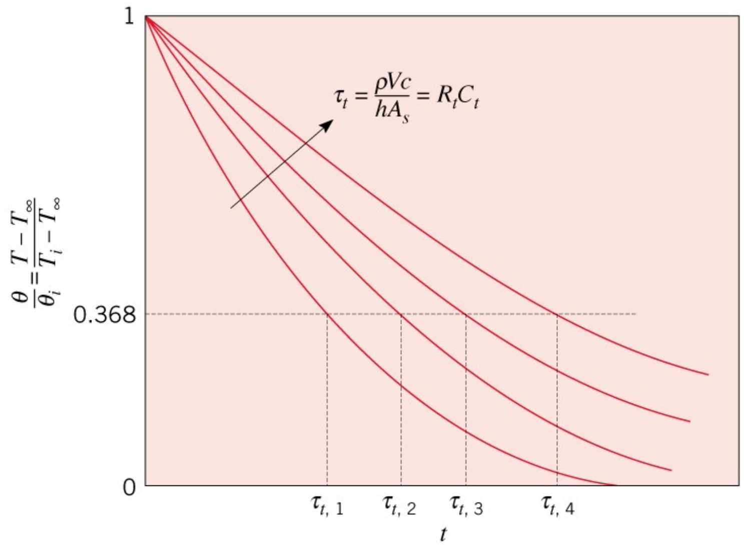
$$\rho \forall c \frac{dT}{dt} = -hA_{s,c}\left(T - T_\infty\right) \tag{5.2}$$

$$\frac{\rho \forall c}{hA_{s,c}} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_o^t dt$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-\left(\frac{hA_{s,c}}{\rho \forall c}\right)t\right] = \exp\left[-\frac{t}{\tau_t}\right]$$

The thermal time constant is defined as

$$\tau_t \equiv \underbrace{\left(\frac{1}{hA_{s,c}}\right)}_{\text{Thermal Resistance, } R_t} \underbrace{\left(\rho \forall c\right)}_{\text{Lumped Thermal Capacitance, } C_t} \tag{5.7}$$



The change in thermal energy storage due to the transient process is

$$\Delta E_{st} \equiv -Q = -\int_o^t \dot{E}_{out} dt = -hA_{s,c}\int_o^t \theta dt = -\left(\rho \forall c\right)\theta_i\left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right] \tag{5.8}$$

- Negligible Convection and Source Terms $\left(h_r >> h,\ \dot{E}_g = 0,\ q_s'' = 0 \right)$:

Assuming radiation exchange with large surroundings,

$$\rho \forall c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma \left(T^4 - T_{sur}^4 \right)$$

$$\frac{\varepsilon A_{s,r} \sigma}{\rho \forall c} \int_o^t dt = \int_{T_i}^{T} \frac{dT}{T_{sur}^4 - T^4}$$

$$t = \frac{\rho \forall c}{4 \varepsilon A_{s,r} \sigma T_{sur}^3} \left\{ \ln \left| \frac{T_{sur} + T}{T_{sur} - T} \right| - \ln \left| \frac{T_{sur} + T_i}{T_{sur} - T_i} \right| + 2 \left[\tan^{-1} \left(\frac{T}{T_{sur}} \right) - \tan^{-1} \left(\frac{T_i}{T_{sur}} \right) \right] \right\} \quad (5.18)$$

Result necessitates implicit evaluation of $T(t)$.

The Biot Number and Validity of The Lumped Capacitance Method

- The Biot Number: The first of many dimensionless parameters to be considered.
  - Definition:

$$Bi \equiv \frac{hL_c}{k}$$

$h \rightarrow$ convection or radiation coefficient

$k \rightarrow$ thermal conductivity of the solid

$L_c \rightarrow$ characteristic length of the solid ($\forall / A_s$ or coordinate associated with maximum spatial temperature difference)

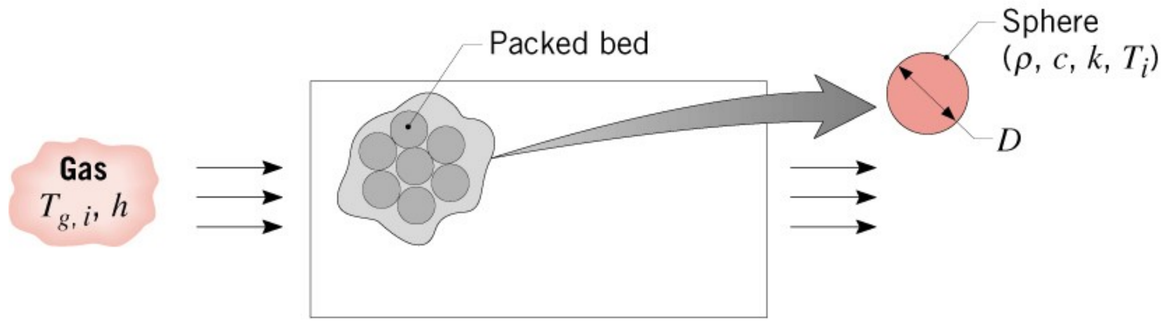
  - Physical Interpretation:

$$Bi = \frac{L_c / kA_s}{1/hA_s} \,\square\, \frac{R}{R_{conv}} \,\square\, \frac{\Delta T}{\Delta T_{solid/fluid}}$$

  - Criterion for Applicability of Lumped Capacitance Method:

$$Bi << 1$$

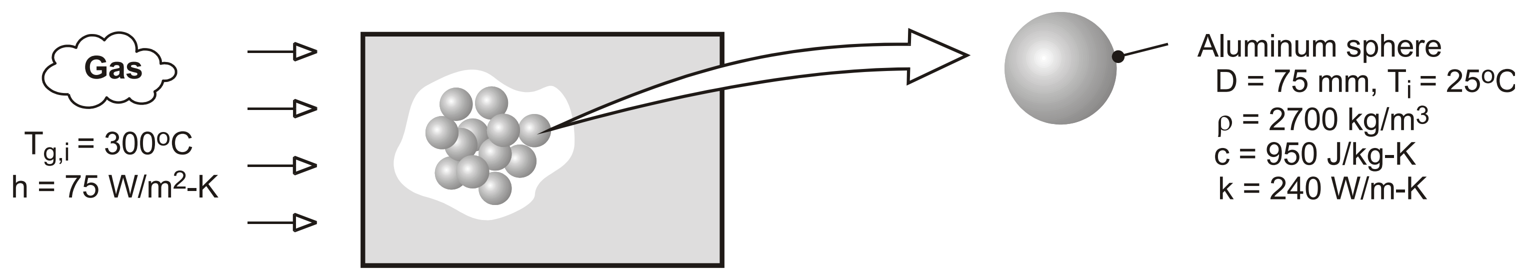
Problem 5.11: Charging a thermal energy storage system consisting of a packed bed of aluminum spheres.



KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere at inlet to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

Schematic:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute $Bi = h(r_o/3)/k = 75\ W/m^2 \cdot K\ (0.025m)/150\ W/m \cdot K = 0.013 << 1$.

Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time.

From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$-\frac{\Delta E_{st}}{\rho c V \theta_i} = 0.90 = 1 - \exp\left(-t/\tau_t\right)$$

$$\tau_t = \rho V c / h A_s = \rho D c / 6h = \frac{2700\ kg/m^3 \times 0.075m \times 950\ J/kg \cdot K}{6 \times 75\ W/m^2 \cdot K} = 427s.$$

$$t = -\tau_t \ln\left(0.1\right) = 427s \times 2.30 = 984s$$

From Eq. (5.6), the corresponding temperature at any location in the sphere is

$$T\left(984s\right) = T_{g,i} + \left(T_i - T_{g,i}\right)\exp\left(-6ht/\rho Dc\right)$$

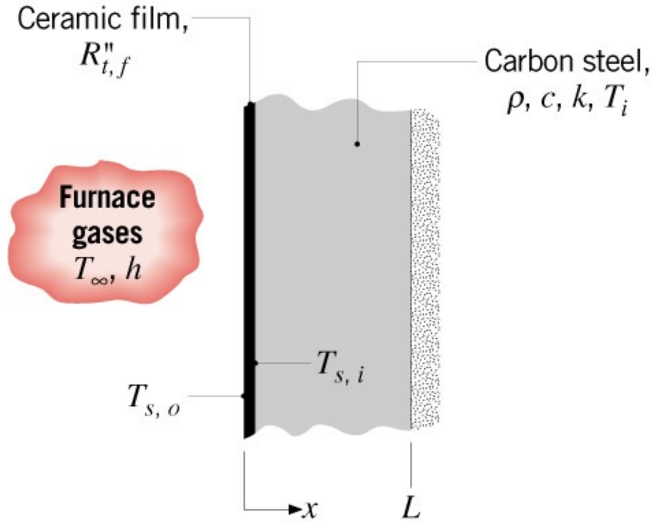
$$T\left(984s\right) = 300°C - 275°C\ \exp\left(-6 \times 75\ W/m^2 \cdot K \times 984s / 2700\ kg/m^3 \times 0.075m \times 950\ J/kg \cdot K\right)$$

$$T\left(984s\right) = 272.5°C$$

If the product of the density and specific heat of copper is $(\rho c)_{Cu} \approx 8900\ kg/m^3 \times 400\ J/kg \cdot K = 3.56 \times 10^6\ J/m^3 \cdot K$, is there any advantage to using copper spheres of equivalent diameter in lieu of aluminum spheres?

Does the time required for a sphere to reach a prescribed state of thermal energy storage change with increasing distance from the bed inlet? If so, how and why?

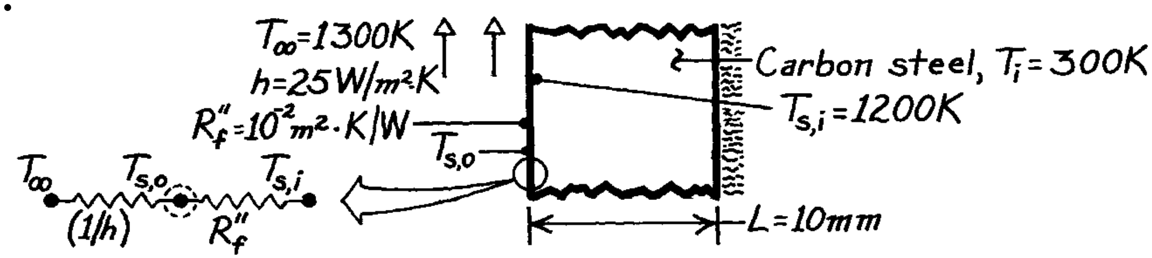
Problem 5.15: Heating of coated furnace wall during start-up.



KNOWN: Thickness and properties of furnace wall. Thermal resistance of ceramic coating on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of coating surface temperature.

Schematic:



ASSUMPTIONS: (1) Constant properties, (2) Negligible coating thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel: $\rho = 7850\ \text{kg/m}^3$, $c = 430\ \text{J/kg·K}$, $k = 60\ \text{W/m·K}$.

ANALYSIS: Heat transfer to the wall is determined by the total resistance to heat transfer from the gas to the surface of the steel, and not simply by the convection resistance.

Hence, with

$$U = \left(R''_{tot}\right)^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \text{W/m}^2\cdot\text{K}} + 10^{-2}\ \text{m}^2\cdot\text{K/W}\right)^{-1} = 20\ \text{W/m}^2\cdot\text{K}.$$

$$Bi = \frac{UL}{k} = \frac{20\ \text{W/m}^2\cdot\text{K} \times 0.01\ \text{m}}{60\ \text{W/m}\cdot\text{K}} = 0.0033 << 1$$

and the lumped capacitance method can be used.

(a) From Eqs. (5.6) and (5.7),

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-t/\tau_t\right) = \exp\left(-t/R_t C_t\right) = \exp\left(-Ut/\rho Lc\right)$$

$$t = -\frac{\rho Lc}{U}\ln\frac{T - T_\infty}{T_i - T_\infty} = -\frac{7850\ \text{kg/m}^3\left(0.01\ \text{m}\right)430\ \text{J/kg}\cdot\text{K}}{20\ \text{W/m}^2\cdot\text{K}}\ln\frac{1200 - 1300}{300 - 1300}$$

$$t = 3886\text{s} = 1.08\text{h}.$$

(b) Performing an energy balance at the outer surface (s,o),

$$h\left(T_{\infty} - T_{s,o}\right) = \left(T_{s,o} - T_{s,i}\right) / R''_f$$

$$T_{s,o} = \frac{hT_{\infty} + T_{s,i} / R''_f}{h + \left(1 / R''_f\right)} = \frac{25\ \mathrm{W/m^2 \cdot K} \times 1300\ \mathrm{K} + 1200\ \mathrm{K} / 10^{-2}\mathrm{m^2 \cdot K/W}}{\left(25 + 100\right)\mathrm{W/m^2 \cdot K}}$$

$$T_{s,o} = 1220\ \mathrm{K}.$$

How does the coating affect the thermal time constant?