Differential Calculus

First derivative $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$,

Second derivative $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$,

...

...

Inductively, $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$.

Successive Differentiation

They are also denoted by $y^{(1)}$, $y^{(2)}$,…, $y^{(n)}$. $y^{(0)}$ denotes y.

2. $y = a^{mx}$ where m is a positive integer.

$$y_1 = D(a^{mx}) = m\, a^{mx} \log a.$$

$$y_2 = m \log a \,.\, (m\, a^{mx} \log a)$$

$$= (m \log a)^2\, a^{mx}$$

$$y_3 = (m \log a)^3\, a^{mx}$$

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$$y_n = (m \log a)^n\, a^{mx}.$$

3. $y = (ax + b)^m$, where m is a positive integer such that m > n .

$y_1 = m(ax + b)^{m-1}.a$

$y_2 = m(m - 1)(ax + b)^{m-2}. a^2$

$y_3 = m(m - 1) (m- 2) (ax + b)^{m-3} a^3$

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$y_n = m(m- 1) (m - 2) \dots [m - (n - 1)] (ax + b)^{m-n} a^n.$

4. $y = \dfrac{1}{ax + b}$

Let us write $y = (ax + b)^{-1}$

$y_1 = -1\,(ax + b)^{-2}.a = (-1)^1\,1!\,(ax + b)^{-2}.a$

$y_2 = (-1)\,(-2)\,(ax + b)^{-3}\,a^2 = (-1)^2\,2!\,(ax + b)^{-3}\,.\,a^2$

$y_3 = (-1)\,(-2)\,(-3)\,(ax + b)^{-4}\,a^3 = (-1)^3\,3!\,(ax + b)^{-4}\,a^3$

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$y_n = (-1)^n\,n!\,(ax + b)^{-(n+1)}\,a^n$

$$y_n = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}}$$

5. $y = \log(ax + b)$

$y_1 = a(ax + b)^{-1}$

$y_2 = a(-1)(ax + b)^{-2}.a = a^2(-1)^1\, 1!\, (ax + b)^{-2}$

$y_3 = a^2(-1)(-2)(ax + b)^{-3}\ .a = a^3(-1)^2\, 2!\, (ax + b)^{-3}$

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$y_n = a^n(-1)^{n-1}(n - 1)!\,(ax + b)^{-n}$

$$y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

Exercise: If $y = e^{ax} \cos(bx + c)$, $y_n = r^n e^{ax} \cos(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $q = \tan^{-1}(b/a)$.

Examples:

1. Find the n^{th} derivative of $y = \cos h^2 3x$

Solution: Write $\cos h^2 3x = \frac{1}{4}(e^{6x} + e^{-6x} + 2)$

$$y_n = \frac{1}{4}[6^n e^{6x} + (-6)^n e^{-6x}].$$

Find the n^{th} derivative of : (1) sin h 2x sin 4x

Solution: $D^n[\sinh 2x \sin 4x]$

$$= \frac{1}{2}(D^n\ [e^{2x} \sin 4x] - D^n\ [e^{-2x}\ \sin 4x])$$

$$= \frac{1}{2} 20^{n/2}\{e^{2x}\ \sin(4x + n \tan^{-1} 2) - e^{-2x}\ \sin(4x - n \tan^{-1} 2)\}$$

(2) $y = \log(4x^2 - 1)$

Solution: Let $y = \log(4x^2 - 1) = \log[(2x + 1)(2x - 1)]$

Therefore $y = \log(2x + 1) + \log(2x - 1)$.

$$y_n = \frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n} + \frac{(-1)^{n-1}(n-1)!2^n}{(2x-1)^n}$$

Find the nth derivative of $y = \dfrac{x^2}{(x+2)(2x+3)}$

Solution: $y = \dfrac{x^2}{(x+2)(2x+3)} = \dfrac{x^2}{2x^2+7x+6}$

$$= \frac{\frac{1}{2}\left(2x^2+7x+6\right) - \frac{1}{2}(7x+6)}{2x^2+7x+6} = \frac{1}{2} - \frac{1}{2}\frac{(7x+6)}{(x+2)(2x+3)}$$

$$= \frac{1}{2}\left[1 - \frac{A}{x+2} + \frac{B}{2x+3}\right] = \frac{1}{2}\left[1 - \frac{8}{x+2} + \frac{9}{2x+3}\right]$$

$$\frac{d^n}{dx^n}\left[\frac{x^2}{(x+2)(2x+3)}\right] = -\frac{8(-1)^n n!}{2(x+2)^{n+1}} + \frac{9(-1)^n n! 2^n}{2(2x+3)^{n+1}}$$