Limit and Continuity

The Limit Process

THE LIMIT PROCESS (AN INTUITIVE INTRODUCTION)

We could begin by saying that limits are important in calculus, but that would be a major understatement. *Without limits, calculus would not exist. Every single notion of calculus is a limit in one sense or another*.

For example:

What is the slope of a curve? It is the limit of slopes of secant lines.

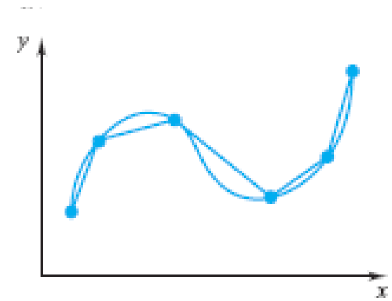

Figure 2.1.2

What is the length of a curve? It is the limit of the lengths of polygonal paths inscribed in the curve.

Salas, Hille, Etgen Calculus: One and Several Variables

The Limit Process

Example 1
Set $f(x) = 4x + 5$ and take $c = 2$. As x approaches 2, $4x$ approaches 8 and $4x + 5$ approaches $8 + 5 = 13$. We conclude that

$$\lim_{x \to 2} f(x) = 13.$$

The Limit Process

Example 2

Set

$$f(x) = \sqrt{1-x} \text{ and take } c = -8.$$

As x approaches −8, 1 − x approaches 9 and $\sqrt{1-x}$ approaches 3. We conclude that

$$\lim_{x \to -8} f(x) = 3$$

If for that same function we try to calculate

$$\lim_{x \to 2} f(x)$$

we run into a problem. The function $f(x) = \sqrt{1-x}$ is defined only for $x \leq 1$. It is therefore not defined for x near 2, and the idea of taking the limit as x approaches 2 makes no sense at all:

$$\lim_{x \to 2} f(x) \text{ does not exist.}$$

The Limit Process

Example 3

$$\lim_{x \to 3} \frac{x^3 - 2x + 4}{x^2 + 1} = \frac{5}{2}.$$

First we work the numerator: as x approaches 3, x^3 approaches 27, $-2x$ approaches –6, and $x^3 - 2x + 4$ approaches 27 – 6 + 4 = 25. Now for the denominator: as x approaches 3, $x^2 + 1$ approaches 10. The quotient (it would seem) approaches 25/10 = 52.

Continuity

Continuity at a Point

The basic idea is as follows: We are given a function *f* and a number *c*. We calculate (if we can) both $\lim_{x \to c} f(x)$ and *f* (*c*). If these two numbers are equal, we say that *f* is *continuous* at *c*. Here is the definition [illegible] stated.

> **DEFINITION 2.4.1**
>
> Let f be a function defined at least on an open interval $(c - p, c + p)$. We say that *f is continuous at c* if
>
> $$\lim_{x \to c} f(x) = f(c).$$

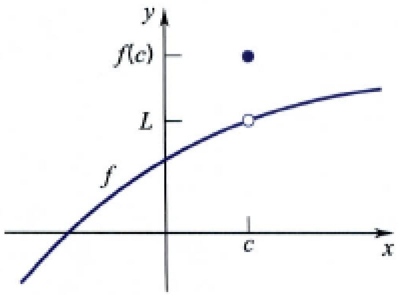

Figure 2.4.1

If the domain of *f* contains an interval (*c* − *p, c* + *p*), then *f* can fail to be continuous at *c* for only one of two reasons: either

(i) *f has a limit as x tends to c, but* $\lim_{x \to c} f(x) \neq f(c)$, *or*

(ii) *f has no limit as x tends to c*.

In case (i) the number *c* is called a *removable* discontinuity. The discontinuity can be removed by redefining *f* at *c*. If the limit is *L*, redefine *f* at *c* to be *L*.

In case (ii) the number c is called an *essential* discontinuity. You can change the value of *f* at a billion points in any way you like. The discontinuity will remain.

Continuity

The functions shown have essential discontinuities at *c*.

The discontinuity in Figure 2.4.2 is, for obvious reasons, called a ***jump*** discontinuity.

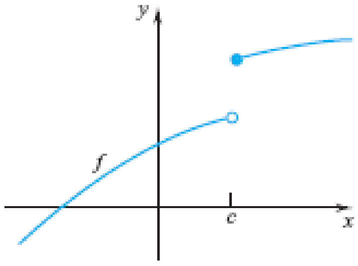

Figure 2.4.2

The functions of Figure 2.4.3 have ***infinite*** discontinuities.

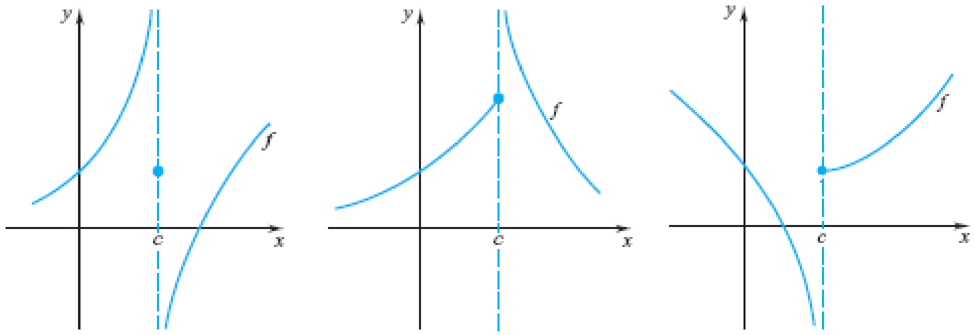

Figure 2.4.3

Continuity

THEOREM 2.4.2

If f and g are continuous at c, then

(i) $f + g$ is continuous at c;
(ii) $f - g$ is continuous at c;
(iii) αf is continuous at c for each real α;
(iv) $f \cdot g$ is continuous at c;
(v) f/g is continuous at c provided $g(c) \neq 0$.

Continuity

Example 1

The function

$$F(x) = 3|x| + \frac{x^3 - x}{x^2 - 5x + 6} + 4$$

is continuous at all real numbers other than 2 and 3. You can see this by noting that

$$F = 3f + g/h + k$$

where

$$f(x) = |x|, \quad g(x) = x^3 - x, \quad h(x) = x^2 - 5x + 6, \quad k(x) = 4.$$

Since *f, g, h, k* are everywhere continuous, *F* is continuous except at 2 and 3, the numbers at which *h* takes on the value 0. (At those numbers *F* is not defined.)

Continuity

Example 2

The function $F(x) = \sqrt{\dfrac{x^2 + 1}{x - 3}}$ is continuous at all numbers greater than 3. To see this,

note that $F = f \circ g$, where

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \frac{x^2 + 1}{x - 3}.$$

Now, take any $c > 3$. Since *g* is a rational function and *g* is defined at *c*, *g* is continuous at *c*. Also, since *g*(*c*) is positive and f is continuous at each positive number, *f* is continuous at *g*(*c*). By Theorem 2.4.4, *F* is continuous at *c*.

Continuity

Example 3

The function $F(x) = \dfrac{1}{5 - \sqrt{x^2 + 16}}$ is continuous everywhere except at x = ±3,

where it is not defined. To see this, note that $F = f \circ g \circ k \circ h$, where

$$f(x) = \frac{1}{x}, \qquad g(x) = 5 - x, \qquad k(x) = \sqrt{x}, \qquad h(x) = x^2 + 16.$$

and observe that each of these functions is being evaluated only where it is continuous. In particular, *g* and *h* are continuous everywhere, *f* is being evaluated only at nonzero numbers, and *k* is being evaluated only at positive numbers. .

Continuity

Example 4

Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} 2x + 1, & x \leqq 0 \\ 1, & 0 < x \leqq 1 \\ x^2 + 1, & x > 1. \end{cases}$$

(Figure 2.4.8)

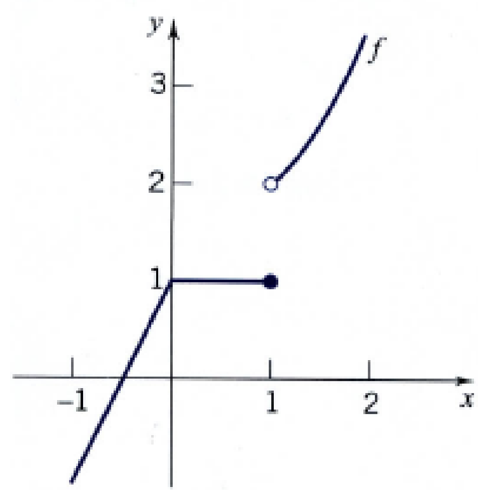

Figure 2.4.8

Continuity

Example 5

Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} x^3, & x \leqq -1 \\ x^2 - 2, & -1 < x < 1 \\ 6 - x, & 1 \leqq x < 4 \\ \dfrac{6}{7 - x}, & 4 < x < 7 \\ 5x + 2, & x \geqq 7. \end{cases}$$