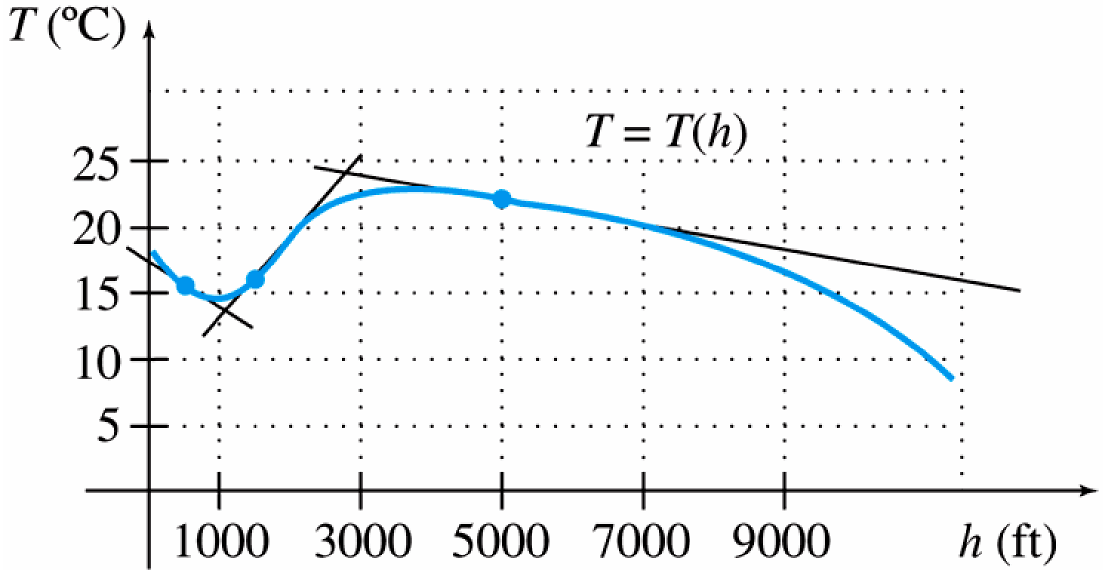
Total Derivative

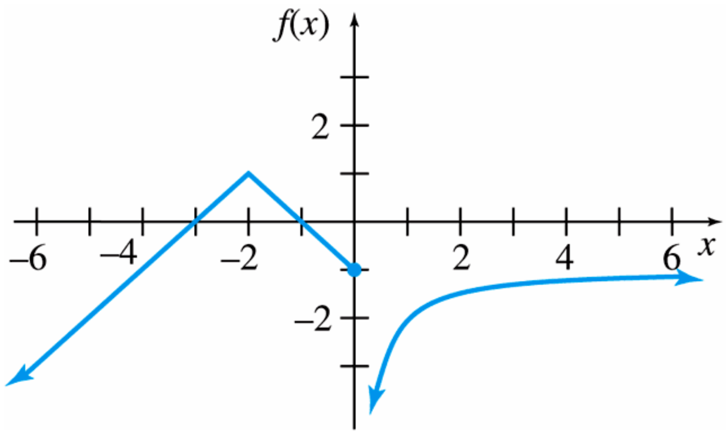
Graphing the Derivative

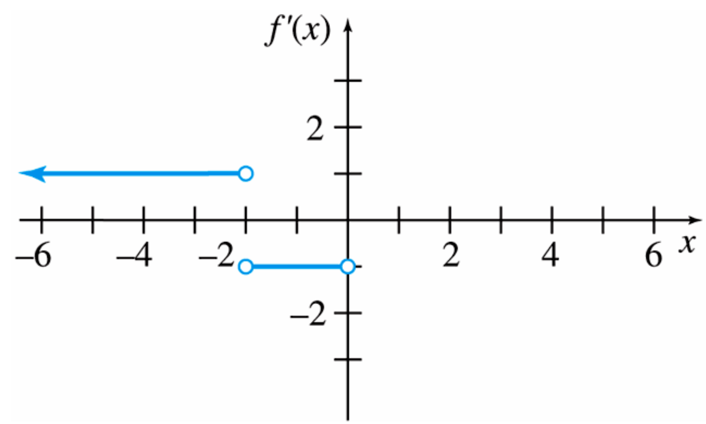
- When graphing the derivative, you are graphing the slope of the original function.



Graphing the Derivative

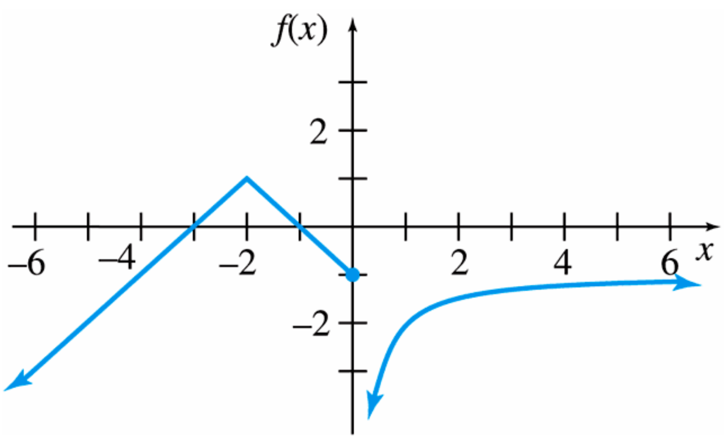
- When x < -2, the slope is 1
- When -2 < x < 0, the slope is -1
- At x = -2 and x = 0 the derivative does not exist—why?

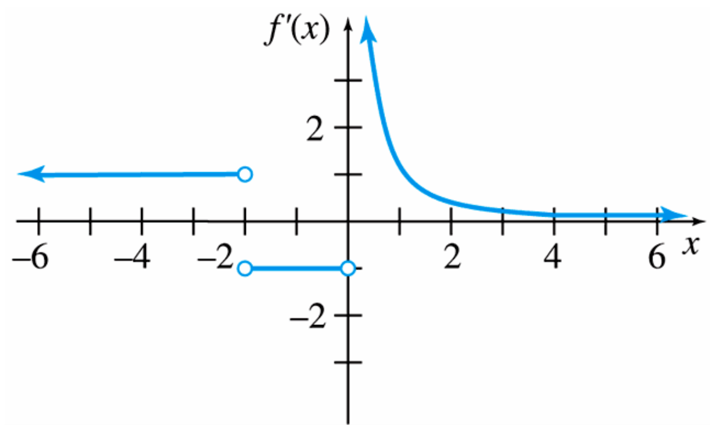




Graphing the Derivative

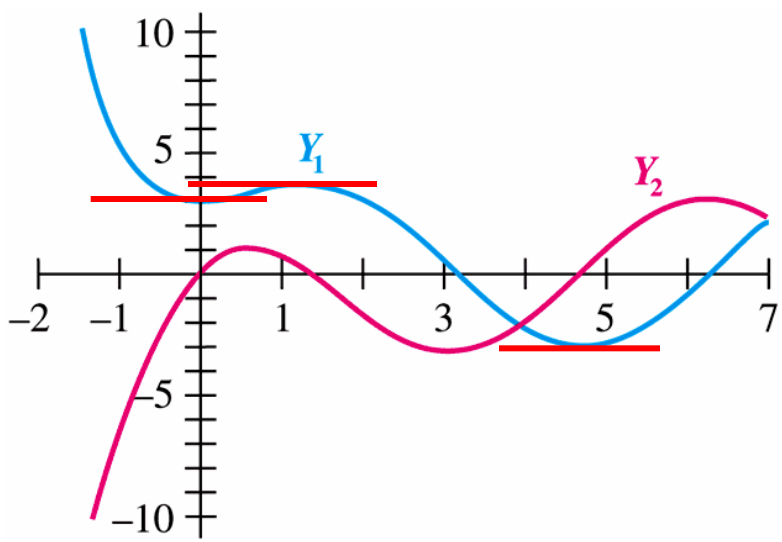
- For x > 0, the derivative is positive—estimated to be a slope of 1 at x = 1
- As x approaches 0 from the right, the derivative becomes larger
- As x approaches infinity, the derivative approaches 0.





Graphing

- Which is the f(x) and which is f'(x)?
- The derivative is 0 (crosses the x-axis) wherever there is a horizontal tangent
- Y1 = f(x)
- Y2 = f'(x)



Calculating the Derivative

Notation

NOTATIONS FOR THE DERIVATIVE

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}[f(x)], \qquad \text{or} \qquad D_x[f(x)].$$

Constant Rule

CONSTANT RULE

If $f(x) = k$, where k is any real number, then

$$\boldsymbol{f'(x) = 0}.$$

(The derivative of a constant is 0.)

If f(x) = 4, then f'(x) = 0

If f(x) = π, then f'(x) = 0

Power Rule

POWER RULE

If $f(x) = x^n$ for any real number n, then

$$\boldsymbol{f'(x) = nx^{n-1}}.$$

(The derivative of $f(x) = x^n$ is found by multiplying by the exponent n and decreasing the exponent on x by 1.)

Power Rule – Examples

- If $f(x) = x^6$, find $D_x y$
- $D_x y = 6x^{6-1} = 6x^5$
- If $f(x) = x, \text{find } \dfrac{dy}{dx}$

$$\frac{dy}{dx} = 1x^0 = 1$$

$\dfrac{1}{x^3}$ must be rewritten

x^{-3}

- If $y = \dfrac{1}{x^3} \text{ find } \dfrac{dy}{dx}$

$$\frac{dy}{dx} = -3x^{-4} = \frac{-3}{x^4}$$

Power Rule Examples

- **Example 1:** Given $f(x) = 3x^2$, find $f'(x)$.
- $f'(x) = 6x$
- **Example 2:** Find the first derivative given $f(x) = 8x$.
- $8x^0 = 8$

Sum or Difference Rule

SUM OR DIFFERENCE RULE

If $f(x) = u(x) \pm v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u'(x) \pm v'(x).}$$

(The derivative of a sum or difference of functions is the sum or difference of the derivatives.)

Sum/Difference Examples

- The Sum/Difference rule can be used on each term in a polynomial to find the first derivative.
- Find f'(x), given $f(x) = 5x^4 – 2x^3 – 5x^2 + 8x + 11$
- $f'(x) = 20x^3 – 6x^2 – 10x + 8$
- The derivative of a constant is 0 because 11 is the same as $11x^0$, which is $(0)11x^{-1}$

Sum/Difference Examples

- Find p'(t) given $p(t) = 12t^4 - 6\sqrt{t} = \frac{5}{t}$
- Rewrite p(t): $p(t) = 12t^4 - 6t^{\frac{1}{2}} + 5t^{-1}$

$$p'(t) = 48t^3 - 3t^{-\frac{1}{2}} - 5t^{-2}$$

$$p'(t) = 48t^3 - \frac{3}{\sqrt{t}} - \frac{5}{t^2}$$

Applications

- Marginal variables can be cost, revenue, and/or profit. Marginal refers to rates of change.
- Since the derivative gives the rate of change of a function, we find the derivative.

Application Example

- The total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by
- $C(x) = 4x^2 + 100x + 500$
- Find the marginal cost for $x = 5$

- $C'(x) = 8x + 100$; $C'(5) = 140$

Example Continued

- After 5,000 barrels have been produced, the cost to produce 1,000 more barrels will be approximately $14,000
- The actual cost will be C(6) – C(5): 144 or $14,400

Product Rule

PRODUCT RULE

If $f(x) = u(x) \cdot v(x)$, and if $u'(x)$ and $v'(x)$ both exist, then

$$\boldsymbol{f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)}.$$

(The derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.)

Product Rule - Example

- Let $f(x) = (2x + 3)(3x^2)$. Find $f'(x)$

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

- $=(2x + 3)(6x) + (3x^2)(2)$
- $=12x^2 + 18x + 6x^2 = 18x^2 + 18x$

Power Rule

- Find f'(x) given that $f(x) = \left(\sqrt{x} + 3\right)\left(x^2 - 5x\right)$

$$\left(x^{\frac{1}{2}} + 3\right)(2x - 5) + \left(x^2 - 5x\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$\frac{5}{2}x^{\frac{3}{2}} + 6x - \frac{15}{2}x^{\frac{1}{2}} - 15$$

Quotient Rule

QUOTIENT RULE

If $f(x) = u(x)/v(x)$, if all indicated derivatives exist, and if $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

(The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.)

Quotient Rule Example

- Find f'(x) if $f(x) = \frac{2x-1}{4x+3}$

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

$$= \frac{(4x+3)(2) - (2x-1)(4)}{(4x+3)^2}$$

$$= \frac{10}{(4x+3)^2}$$

Product & Quotient Rules

- Find $$D_x\left[\frac{(3-4x)(5x+1)}{7x-9}\right]$$

$$\frac{(7x-9)D_x\left[(3-4x)(5x+1)\right]-\left[(3-4x)(5x+1)\right]D_x(7x-9)}{(7x-9)^2}$$

$$\frac{(7x-9)\left[(3-4x)(5)+(5x+1)(-4)\right]-(3+11x-20x^2)(7)}{(7x-9)^2}$$

$$\frac{-140x^2+360x-120}{(7x-9)^2}$$

Partial differentiation

Definition of the partial derivative

- the partial derivative of f(x,y) with respect to x and y are

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \left(\frac{\partial f}{\partial x}\right)_y = f_x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \left(\frac{\partial f}{\partial y}\right)_x = f_y$$

- for general n-variable

$$\frac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots x_i + \Delta x_i, \ldots x_n) - f(x_1, x_2, \ldots x_i, \ldots x_n)}{\Delta x_i}$$

- second partial derivatives of two-variable function f(x,y)

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Partial differentiation

The total differential and total derivative

$$x \to x + \Delta x \text{ and } y \to y + \Delta y \Rightarrow f \to f + \Delta f$$

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)$$

$$= \left[\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\right]\Delta x + \left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\right]\Delta y$$

as $\Delta x \to 0$ and $\Delta y \to 0$, the <u>total differential</u> df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

for n - variable function $f(x_1, x_2, \ldots x_n)$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x_n}dx_n$$

variables x_i, $i = 1,2,3,\ldots\ldots, n$, for a given $x_i = x_i(x_1)$

the <u>total derivative</u> of $f(x_1, x_2, \ldots x_i \ldots x_n)$ with respect to x_1 is

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \left(\frac{\partial f}{\partial x_2}\right)\frac{dx_2}{dx_1} + \ldots\ldots\ldots + \left(\frac{\partial f}{\partial x_i}\right)\frac{dx_i}{dx_1} + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)\frac{dx_n}{dx_1}$$

Partial differentiation

Ex: Find the total derivative of $f(x, y) = x^2 + 3xy$ with respect to x, given that $y = \sin^{-1} x$

$$\frac{\partial f}{\partial x} = 2x + 3y, \quad \frac{\partial f}{\partial y} = 3x; \quad \frac{dy}{dx} = \frac{1}{(1 - x^2)^{1/2}}$$

$$\frac{df}{dx} = 2x + 3y + 3x\frac{1}{(1 - x^2)^{1/2}} = 2x + 3\sin^{-1} x + \frac{3x}{(1 - x^2)^{1/2}}$$

The chain rule

for $f = f(x, y)$ and $x = x(u), \; y = y(u)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \Rightarrow \frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

for many variables $f(x_1, x_2, \ldots, x_n)$ and $x_i = x_i(u)$

$$\Rightarrow \frac{df}{du} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{du} = \frac{\partial f}{\partial x_1}\frac{dx_1}{du} + \frac{\partial f}{\partial x_2}\frac{dx_2}{du} + \ldots\ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{du}$$

Partial differentiation

Change of variables

$$f = f\,(x_1, x_2, \ldots, x_n) \text{ and } x_i = x_i\,(u_1, u_2, \ldots, u_m)$$

$$\Rightarrow \frac{\partial f}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial u_j} \qquad j = 1,2,\ldots,m$$

Ex: Polar coordinates ρ and ψ, Cartesian coordinates x and y, x=ρcosφ, y=ρsinφ, $f\,(x, y) \to g(\rho, \phi)$ transform $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ into one in ρ and φ

$$\rho^2 = x^2 + y^2, \quad \frac{\partial \rho}{\partial x} = \frac{x}{(x^2 + y^2)^{1/2}} = \cos\phi, \ \frac{\partial \rho}{\partial y} = \sin\phi$$

$$\phi = \tan^{-1}\ (y/x), \ \frac{\partial \varphi}{\partial x} = \frac{-y/x^2}{1 + (y/x)^2} = \frac{-y}{x^2 + y^2} = \frac{-\rho\sin\phi}{\rho^2} = \frac{-\sin\phi}{\rho}, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{\rho}$$

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{ and } \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \Rightarrow \nabla^2 f\,(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 g}{\partial \phi^2}$$