TRACING OF CURVES

- Given the equation of a curve explicitly as $y = f(x)$ or implicitly as $g(x,y) = c$, a constant, many properties of the curve can be determined easily by knowing its graph. Here we will study the method of tracing a curve whose equation is given in cartesian, polar or parametric equations .

1. Symmetry

- Find out whether the curve is symmetric about any line or a point. The various kinds of symmetry arising from the form of the equation are as follows:

- i) If the equation of the curve is an even function of x, then the curve is symmetric about the y-axis.

- ii) If the equation of the curve is an even function of y, then the curve is symmetric about x-axis.

- iii) If the equation of the curve is an odd function of x, then the curve is symmetric about the origin.

- iv) If the equation of the curve is unaltered when x and y are interchanged, then the curve is symmetric about the line $y = x$.

- v) The curve is symmetrical about the line $y = -x$ if the equation of the curve remains unaltered when x and y are replaced by $-y$ and $-x$ respectively.

vi) The curve is symmetrical in the opposite quadrant if the equation of the curve remains unaltered when x and y are replaced by –x and –y respectively.

2. Region Of Existence

Find the region for x for which y is well defined and real or the region for y for which x is real.

3. Point of Intersection with the co-ordinate axes and the line of symmetry

- Substitute $x=0$ in the equation of the curve and solve for y and substitute $y = 0$ and solve for x.

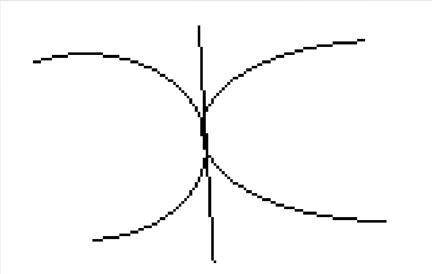
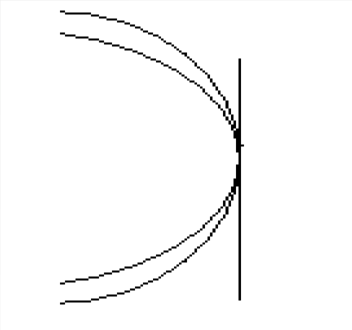
- If the curve is symmetric about the line $y = mx + c$, find the points where the curve cuts the line.

4. Tangents at the origin and at other points

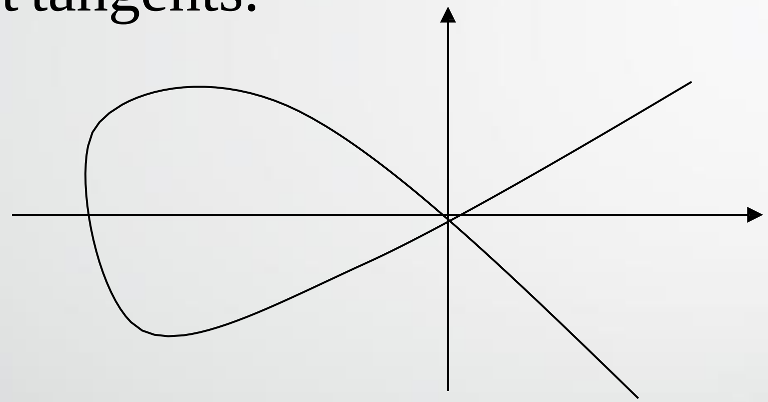
- If more than one branch of the curve passes through a point, then that point is called a multiple point of the curve.

- A double point is a point through which two branches of the curve passes.

- A double point is called a cusp if two branches of the curve have the same tangents.

- A double point is called a node if two branches of the curve have distinct tangents.



- An isolated point of a curve is a point such that there exists a neighborhood of that point in which no other point of the neighborhood lies in the region of existence.

Example:

Consider the curve defined by the equation

$y = x^2(x-2)$

At $x = 0$, $y = 0$, the point (0, 0) lies on the curve

But

$$y = \pm x\sqrt{(x-2)}$$

$\Rightarrow$ y is defined if $x - 2 \geq 0$ or $x \geq 2$

Therefore (0, 0) is an isolated point for this curve.

- The equations of the tangents to the curve at the origin is obtained by equating the lowest degree terms in x and y in the given equation to zero, provided the curve passes through the origin.

- The equations of the tangents to the curve at any point can be determined by shifting the origin to that point and then by equating to zero the lowest degree terms in x and y.

5. Asymptotes

A straight line is said to be an asymptote to an infinite branch of a curve if the perpendicular distance from a point on the curve to the given line approaches to zero as the point moves to infinity along the branch of curve.

- The asymptotes parallel to x-axis are called horizontal asymptotes, those which are parallel to y-axis are called vertical asymptotes and those which are neither parallel to x-axis nor parallel to y-axis are called oblique asymptotes.

- The equations of a vertical asymptotes are obtained by equating the coefficient of highest degree term in y to zero if it is not a constant.

- To obtain the equations of oblique asymptotes, substitutes $y = mx + c$ in the given equation. then equate the coefficients of the highest degree term in x and next highest degree term in x to zero, if it is not a constant, to determine m and c. If the values of m and c exists, then $y = mx + c$ is the equation of the oblique asymptote.

Example:

Consider the curve given by the equation

$$y^2 = x^2((x+a)/(x-a))$$

Or $\quad y^2(x-a) = x^2(x+a)$

The coefficient of highest degree term in x, the coeff. of x^3 is 1 , which is a constant . Hence no horizontal asymptotes. The coefficient of highest degree term in y, the coeff. of y^2 is $x-a$.

Then $x-a = 0$ or $x = a$ is the vertical asymptote.

Put $y = mx + c$ in the given equation, then

$(mx + c)^2(x - a) = x^2(x + a)$

Coefficient of x^3 is $m^2 - 1 = 0$ implies $m^2 = 1$ or $m = \pm 1$

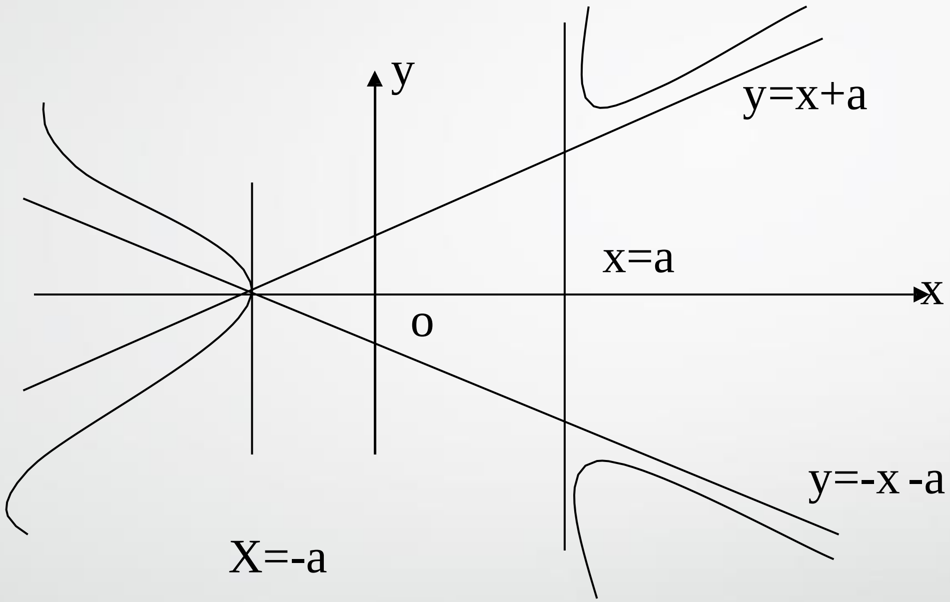
Coefficient of x^2 is $-m^2a + 2mc + a = 0$

$m^2 = 1$ implies $2mc - 2a = 0$.

Therefore $mc = a$.

Then $m = \pm 1$ implies $c = \pm a$

Therefore $y = x + a$ or $y = - x – a$ are the oblique asymptotes.



Graph of $y^2(x-a) = x^2(x+a)$

6.Derivatives

- For the equation of the curve, determine dy/dx

- If dy/dx > 0 in an interval then the curve increases in that interval.
- If dy/dx < 0, then the curve decreases.
- If dy/dx = 0, at a point, then the curve has a stationary point .

- If $d^2y/dx^2 > 0$, in an interval then the curve is concave upwards or convex downwards in that interval.

- If $d^2y/dx^2 < 0$, in an interval then the curve is concave downwards or convex upwards in that interval.

Example: Trace the curve $y^2(a^2+x^2) = x^2(a^2-x^2)$, $a^2>0$

Solution:

(i) Symmetry: Symmetrical about both the axes.

(ii) Points: Passes through (0, 0), (a, 0) and (-a,0).

Loop between (0, 0) and (a, 0); also between (0, 0) and (-a, 0).

(iii) Tangents: Lowest degree terms $a^2y^2-x^2a^2 = 0$, we get $y = \pm x$, the tangents.

(iv) Region:

$$y=\pm x\sqrt{\frac{a^2-x^2}{a^2+x^2}}$$

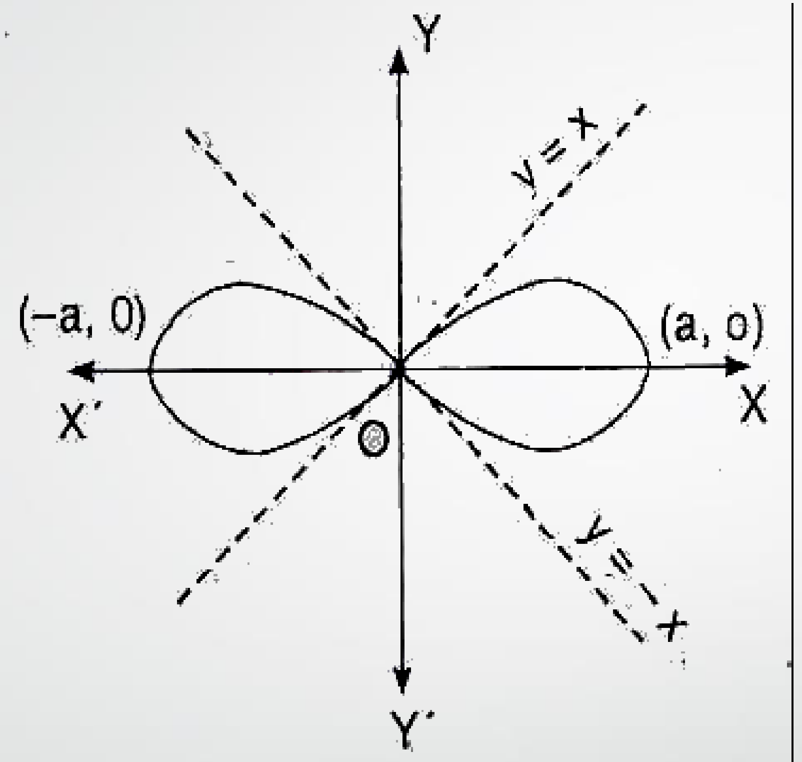
Note that y is imaginary if |x|>a. Hence the whole curve lies between x = a and x = -a.

$$(v)\frac{dy}{dx}=\frac{a^4-2a^2x^2-x^4}{\left(a^2+x^2\right)\left(a^2-x^2\right)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}(a,0)=\infty=\frac{dy}{dx}(-a,0)$$

Therefore the tangents to the curve at (a, 0) and (-a,0) are parallel to Y-axis.

The curve has no asymptotes.



Y
y = x
(−a, 0)
(a, o)
X′
X
O
y = −x
Y′