

Approximations and Errors

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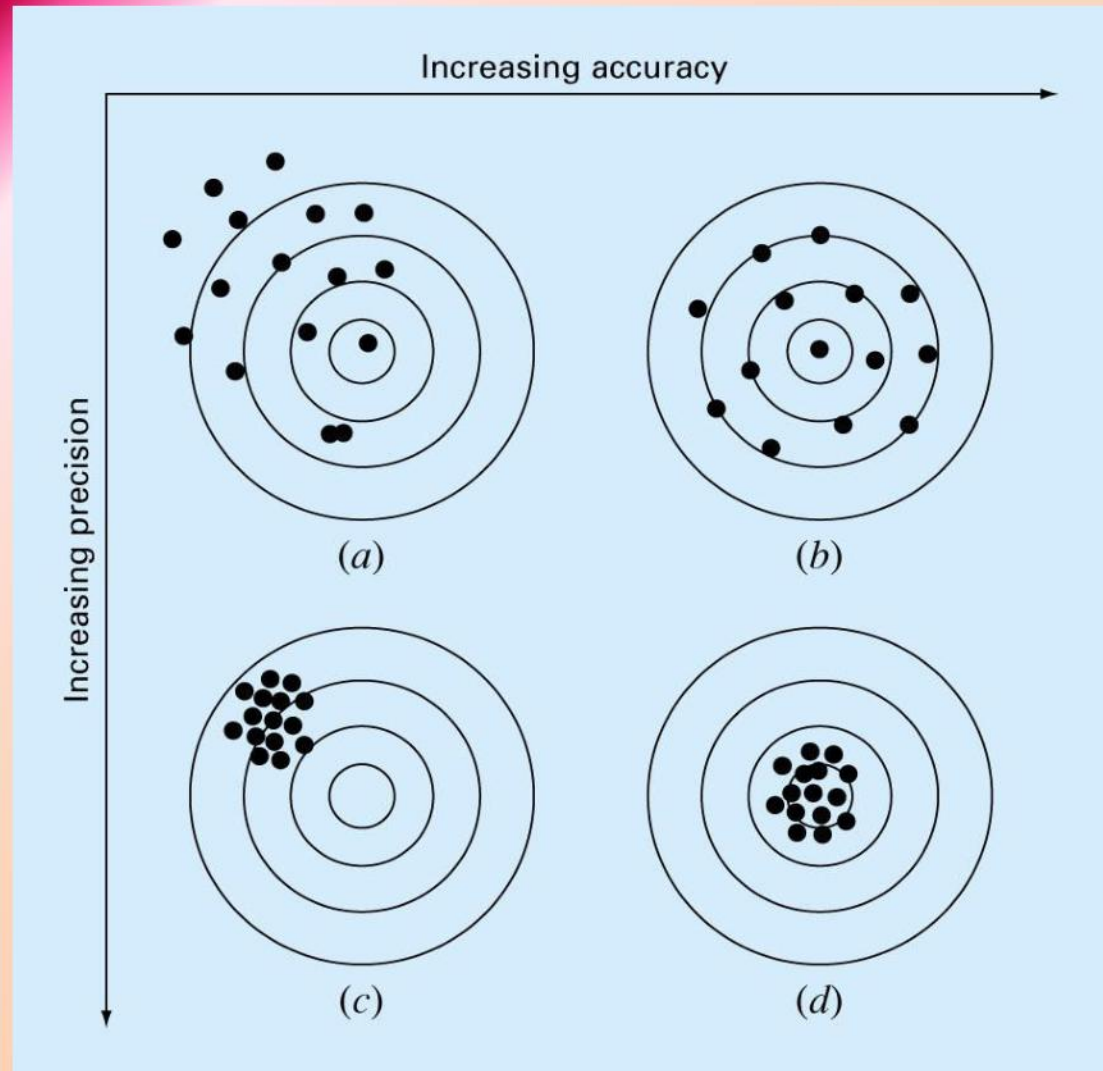
- The major advantage of numerical analysis is that a numerical answer can be obtained even when a problem has no “analytical” solution.
- Although the numerical technique yielded close estimates to the exact analytical solutions, there are errors because the numerical methods involve “approximations”.

Approximations and Round-Off Errors

- For many engineering problems, we cannot obtain analytical solutions.
- Numerical methods yield approximate results, results that are close to the exact analytical solution.
 - Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.
 - Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc ...
 - The output information will then contain error from both of these sources.
- How confident we are in our approximate result?
- The question is “*how much error is present in our calculation and is it tolerable?*”

Accuracy and Precision

- **Accuracy** refers to how closely a computed or measured value agrees with the true value.
- **Precision** refers to how closely individual computed or measured values agree with each other.
- **Bias** refers to systematic deviation of values from the true value.



Significant Figures

Significant figures of a number are those that can be used with confidence.

Rules for identifying sig. figures:

- ***All non-zero digits are considered significant.*** For example, 91 has two significant digits (9 and 1), while 123.45 has five significant digits (1, 2, 3, 4 and 5).
- ***Zeros appearing anywhere between two non-zero digits are significant.*** Example: 101.12 has five significant digits.
- ***Leading zeros are not significant.*** For example, 0.00052 has two significant digits
- ***Trailing zeros are generally considered as significant.*** For example, 12.2300 has six significant digits.

Significant Figures

Scientific Notation

- If it is not clear how many, if any, of zeros are significant. This problem can be solved by using the scientific notation

$$0.0013 = 1.3 * 10^{-3}$$

2 sig. figures

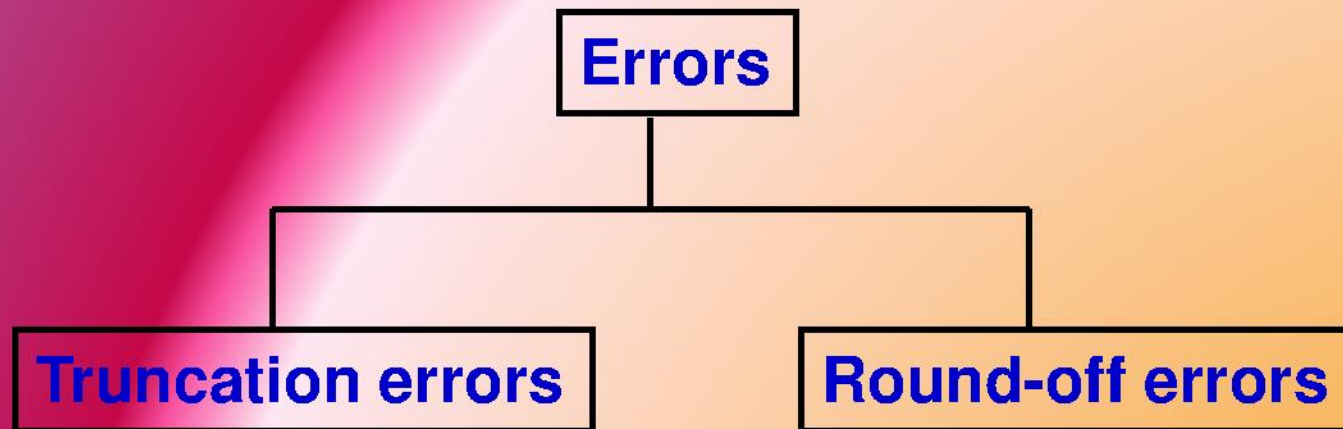
$$0.00130 = 1.30 * 10^{-3}$$

3 sig. figures

- If a number is expressed as $2.55 * 10^4$, (3 s.f), then we are only confident about the first three digits. The exact number may be 25500, 25513, 25522.6 , .. etc. So we are not sure about the last two digits nor the fractional part- If any.
- However, if it is expressed as $2.550 * 10^4$, (4 s.f), then we are confident about the first four digits but uncertain about the last one and the fractional part – if any.

Error Definition

Numerical errors arise from the use of approximations



Result when approximations are used to represent exact mathematical procedure.

Result when numbers having limited significant figures are used to represent exact numbers.

Round-off Errors

- Numbers such as π , e , or $\sqrt{7}$ cannot be expressed by a fixed number of significant figures.
- Computers use a base-2 representation, they cannot precisely represent certain exact base-10 numbers
- Fractional quantities are typically represented in computer using “floating point” form, e.g.,

Example:

$\pi = 3.14159265358$ to be stored carrying 7 significant digits.

$\pi = 3.141592$ **chopping**

$\pi = 3.141593$ **rounding**

Truncation Errors

- Truncation errors are those that result using approximation in place of an exact mathematical procedure.

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{V(t_{i+1}) - V(t_i)}{t_{i+1} - t_i}$$

True Error

□ True error (E_t)

True error (E_t) or Exact value of error
= true value – approximated value

□ True percent relative error (ε_t)

$$\begin{aligned} \text{True percent relative error} = \varepsilon_t &= \frac{\text{True error}}{\text{True value}} \times 100 (\%) \\ &= \frac{\text{true value} - \text{approximated value}}{\text{true value}} \times 100 (\%) \end{aligned}$$

Approximate Error

- The true error is known only when we deal with functions that can be solved analytically.
- In many applications, a prior true value is rarely available.
- For this situation, an alternative is to calculate an **approximation of the error** using the best available estimate of the true value as:

$$\varepsilon_a = \text{Approximate percent relative error} = \frac{\text{Approximate error}}{\text{approximation}} \times 100 (\%)$$

Approximate Error

- In many numerical methods a present approximation is calculated using previous approximation:

$$\varepsilon_a = \frac{\textit{present approximation} - \textit{previous approximation}}{\textit{present approximation}} \times 100 (\%)$$

Note:

- The sign of ε_a or ε_t may be positive or negative
 - We are interested in whether the absolute value is lower than a **prespecified tolerance** (ε_s), not to the sign of error.
- Thus, the computation is repeated until (stopping criteria):

$$|\varepsilon_a| < \varepsilon_s$$

Prespecified Error

- We can relate (ϵ_s) to the number of significant figures in the approximation,

So, we can assure that the result is correct to at least n significant figures if the following criteria is met:

$$\epsilon_s = (0.5 \times 10^{2-n}) \%$$

Example

The exponential function can be computed using Maclaurin series as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Estimate $e^{0.5}$ using series, add terms until the absolute value of approximate error ε_a fall below a pre-specified error ε_s conforming with **three** significant figures.

{The exact value of $e^{0.5}=1.648721\dots$ }

- **Solution**

$$\varepsilon_s = (0.5 \times 10^{2-3})\% = 0.05\%$$

✓ Using one term: $e^{0.5} = 1$ $\varepsilon_t = \frac{1.648721 - 1.0}{1.648721} 100\% = 39.3$

✓ Using two terms: $e^{0.5} = 1 + 0.5 = 1.5$ $\varepsilon_t = \frac{1.648721 - 1.5}{1.648721} 100\% = 9.02\%$ $\varepsilon_a = \frac{1.5 - 1.0}{1.5} 100\% = 33.3\%$

✓ Using three terms: $e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} = 1.625$ $\varepsilon_t = \frac{1.648721 - 1.625}{1.648721} 100\% = 1.44\%$ $\varepsilon_a = \frac{1.625 - 1.0}{1.625} 100\% = 7.69\%$

Terms	Results	$\varepsilon_t\%$	$\varepsilon_a\%$
1	1.0	39.3	---
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.6458333333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158