Lagrange Multipliers

Lagrange Multipliers

Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.77. Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

You want to find x and y such that $f(x, y)$ is a maximum. Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

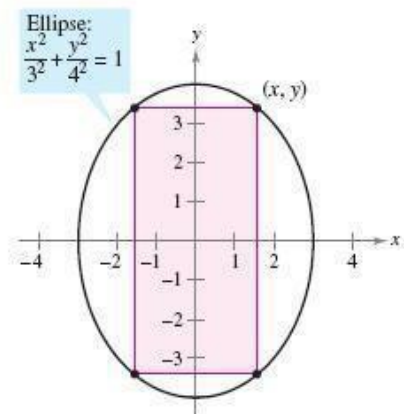


Fig.13.77

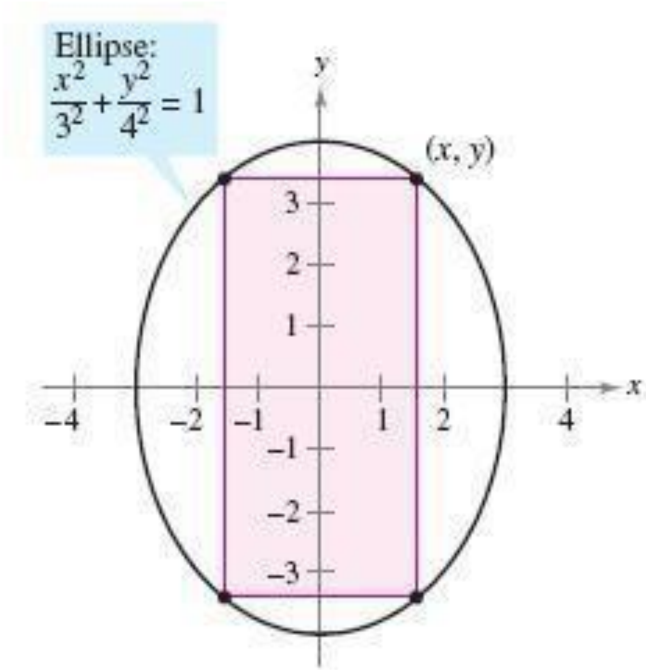
Now, consider the constraint equation to be a fixed level curve of

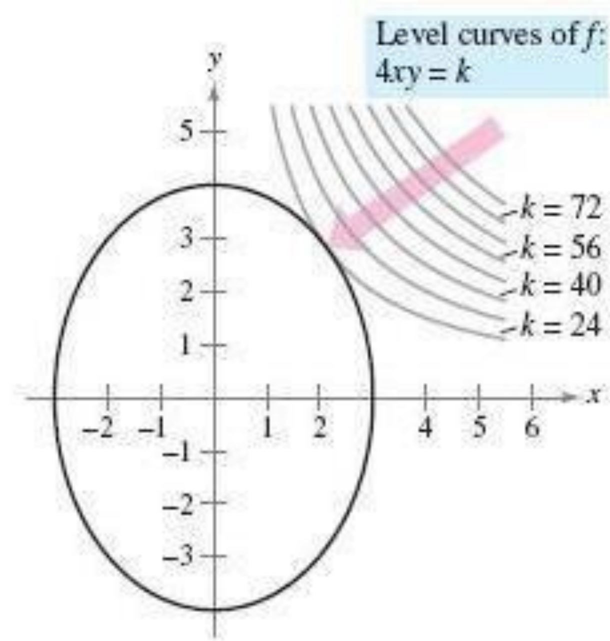
$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

The level curves of f represent a family of hyperbolas

$$f(x, y) = 4xy = k.$$

In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.78.

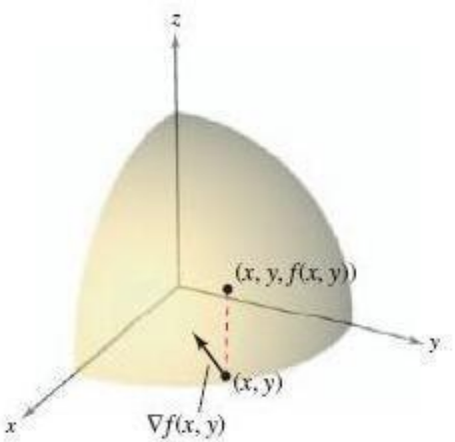




$$Objective\ function : f(x, y) = 4xy$$

$$Constraint : g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

Recall



The gradient of f is a vector in the xy-plane

Fig. 13.48

The Gradient of a Function of Two Variables

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del f." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradient vectors are parallel. This means that $\nabla f(x, y)$ must be a scalar multiple of $\nabla g(x, y)$ at the point of tangency. In the context of constrained optimization problems, this scalar is denoted by λ (the lowercase Greek letter lambda).

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The scalar λ is called a **Lagrange multiplier.** Theorem 13.19 gives the necessary conditions for the existence of such multipliers.

THEOREM 13.19 Lagrange's Theorem

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

The Method of Lagrange Multipliers uses Theorem 13.19 to find the extreme values of a function f subject to a constraint.

Method of Lagrange Multipliers

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

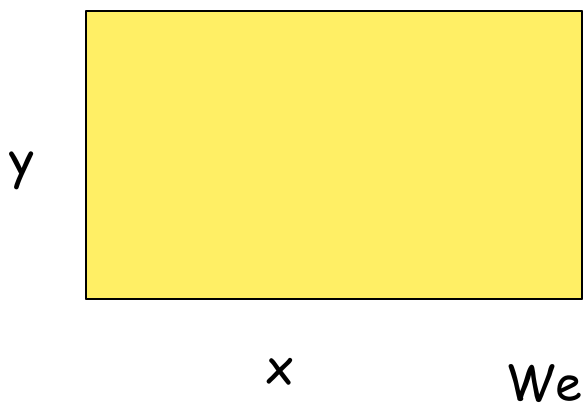
2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

Use Lagrange Multipliers to solve problems

Problem: Suppose I have 50 feet of fencing for a rectangular shaped garden and want to enclose the maximum area.

$2x + 2y = 50$ Constraint

$A = xy$ Objective



We want to get the Area in terms of just x or y so we use the constraint to eliminate one or the other.

$2y=50-2x$

$Y = 25-x$

$A = x(25-x)=25x - x^2$

$A' = 25 - 2x$

$0= 25 - 2x$

$X = 12.5$ and $y = 12.5$

Now we will use the method of **Lagrange Multipliers** to Solve the same problem.

2x + 2y = 50 Constraint
A = xy Objective

f(x,y)=xy is the objective equation and g(x,y)=2x +2y is the constraint equation.

Find

$$f_x(x, y) = y; f_y(x, y) = x$$

$$g_x(x, y) = 2; g_y(x, y) = 2$$

$$\begin{cases} Set\ \ f_x = \lambda g_x \rightarrow y = 2\lambda \\ Set\ \ f_y = \lambda g_y \rightarrow x = 2\lambda \\ 2x + 2y = 50 \end{cases}$$

Solve simultaneously

$$4\lambda + 4\lambda = 50$$

$$8\lambda = 50$$

$$\lambda = 6.25$$

$$x = 12.5$$

$$y = 12.5$$

Now let's look at a problem that might be difficult to solve the old way.

Problem 2: Find the maximum and minimum values for

$f(x, y) = x^2 + y^2$ *objective equation*

$g(x, y) = x^4 + y^4$ *with the* constraint $x^4 + y^4 = 1$

Find $f_x = 2x$; $f_y = 2y$

$g_x = 4x^3$; $g_y = 4y^3$

$$\begin{cases} 2x = \lambda 4x^3 \\ 2y = \lambda 4y^3 \\ x^4 + y^4 = 1 \end{cases}$$

$$2x - \lambda 4x^3 = 2x(1 - \lambda 2x^2) \Rightarrow x = 0 \text{ or } \lambda = 1/2x^2$$

$$2y - \lambda 4y^3 = 2y(1 - \lambda 2y^2) \Rightarrow y = 0 \text{ or } \lambda = 1/2y^2$$

Solve: $1/2x^2 = 1/2y^2 \Rightarrow x^2 = y^2$

If $x = 0$ *then* $y^4 = 1 \Rightarrow y = \pm 1$

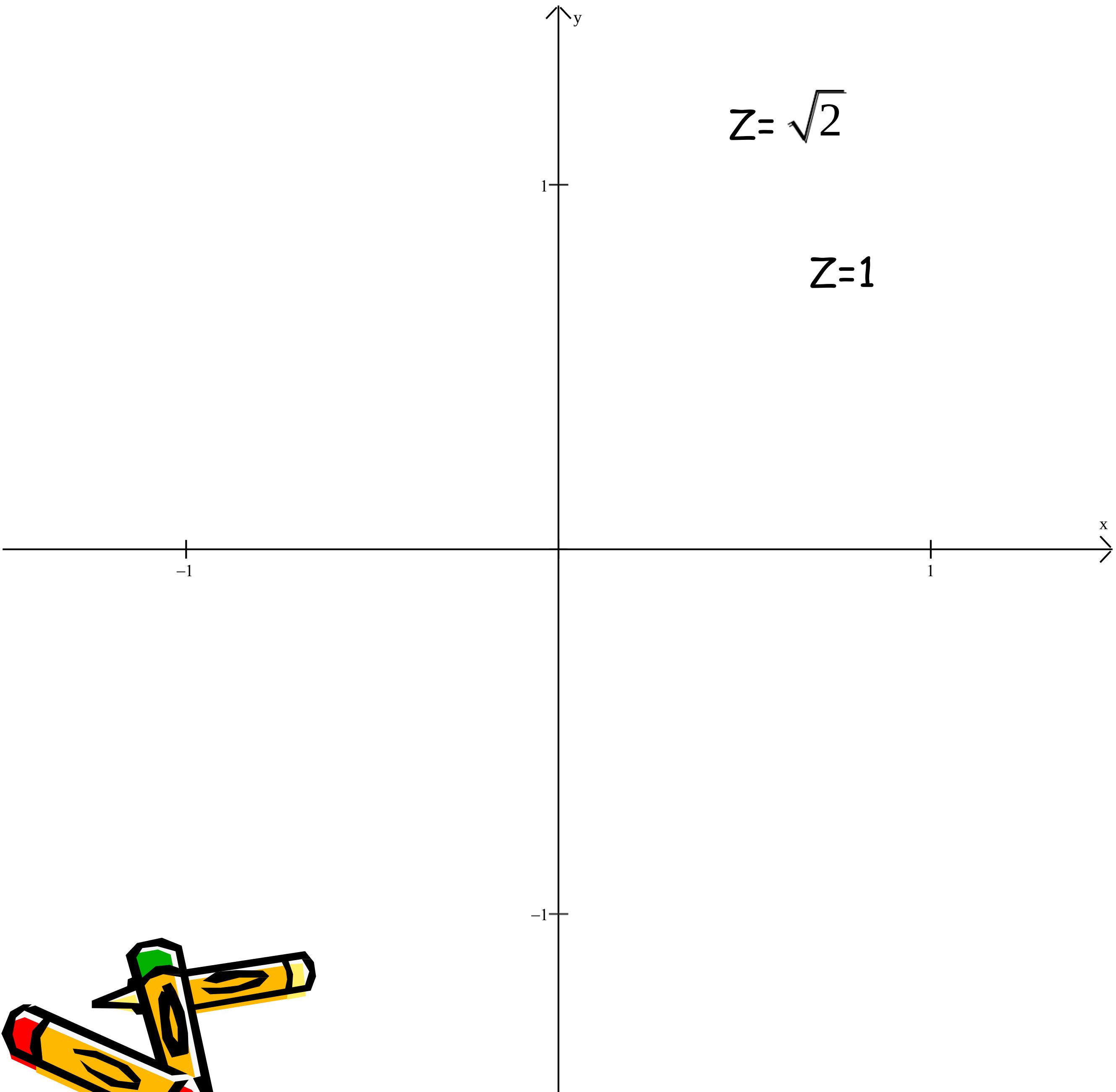
$x^4 + y^4 = 1 \Rightarrow x^4 + x^4 = 1 \Rightarrow 2x^4 = 1$

IIf $y = 0$ *then* $x^4 = 1 \Rightarrow x = \pm 1$

$x = \dfrac{1}{\sqrt[4]{2}}$ *and* $y = \dfrac{1}{\sqrt[4]{2}}$

We could have the following points $\left(\pm\dfrac{1}{\sqrt[4]{2}}, \pm\dfrac{1}{\sqrt[4]{2}}\right)$ *which yield a* max *value of* $\sqrt{2}$

and $(0, \pm 1), (\pm 1, 0)$ *which yields a* min *value of* 1



The constraint is the red graph.

The blue graph and green graph are level curves of the paraboloid.

The blue graph is at 1 and the green graph is at $\sqrt{2}$

which is the exact value for the max.

As the paraboloid extends in the z direction it gets wider than the constraint.

Suppose we want to find the max and min values of f(x,y,z) subject to **two constraints** of the form g(x,y,z)=p and h(x,y,z)= q. Geometrically, this means that we are looking for the extreme values of f when (x,y,z) is restricted to lie on the curve of intersection of the level surfaces g and h. It can be shown that if an extreme value occurs at $\left(x_o, y_o, z_o\right)$, then the gradient vector $\nabla f(x_o, y_o, z_o)$ is in the plane determined by $\nabla g(x_o, y_o, z_o)$ and $\nabla h(x_o, y_o, z_o)$.

We assume these gradient vectors are not 0 or parallel and thus there are numbers $\lambda \; and \; \mu$ (Lagrange multipliers) such that

$$\nabla f(x_o, y_o, z_o) = \lambda \nabla g(x_o, y_o, z_o) + \mu \nabla h(x_o, y_o, z_o)$$

Problem: Maximize f(x,y,z) = xyz subject to the two constraints $x^2 + z^2 = 5 \ and \ x - 2y = 0$

$$f(x,y,z) = xyz \qquad f_x = yz \quad f_y = xz \quad f_z = xy$$

$$g(x,y,z) = x^2 + z^2 \qquad g_x = 2x \quad g_y = 0 \quad g_z = 2z$$

$$h(x,y,z) = x - 2y \qquad h_x = 1 \quad h_y = -2 \quad h_z = 0$$

$$x^2 + z^2 = 5 \ and \ x - 2y = 0$$

Solve

$$yz = \lambda \ 2x \ +$$

$$\mu \ \ xz = \lambda 0 \ -$$

$$\mu 2 \ \ xy = \lambda \ 2z$$

$$+ \mu 0$$

$$x^2 + z^2 = 5 \ and \ x - 2y = 0$$

2

$$xz = -2\mu \Rightarrow \mu = \frac{-xz}{2}$$

$$xy = \lambda 2z \Rightarrow \lambda = \frac{xy}{2z}$$

$$yz = \frac{xy}{2z}2x + \frac{-xz}{2}$$

$$2yz^2 = 2x^2 y - xz^2$$

$$Sub \;\; y = \frac{x}{2} \;\; from \; x - 2y = 0$$

$$Sub \; z^2 = 5 - x^2 \;\; from \; x^2 + z^2 = 5$$

$$2\frac{x}{2}\left(5 - x^2\right) = 2x^2\frac{x}{2} - x\left(5 - x^2\right)$$

$$x\left(5 - x^2\right) = x^2 x - x\left(5 - x^2\right)$$

$$2x\left(5 - x^2\right) = x^3$$

$$10x - 2x^3 - x^3 = 0$$

$$10x - 3x^3 = 0$$

$$x(10 - 3x^2) = 0$$

$$x = 0, \; or \;\; x^2 = \frac{10}{3} \Rightarrow x = \pm\sqrt{\frac{10}{3}}$$

Continued

$$y = \frac{x}{2}$$

$$z^2 = 5 - x^2$$

$$x = 0, \; or \;\; x^2 = \frac{10}{3} \Rightarrow x = \pm\sqrt{\frac{10}{3}}$$

$$for \;\; x = 0, y = 0, z = \pm\sqrt{5}$$

$$for \;\; x = \sqrt{\frac{10}{3}}, \;\; y = \frac{1}{2}\sqrt{\frac{10}{3}},$$

$$z^2 = 5 - \frac{10}{3} = \frac{5}{3}, \; z = \pm\sqrt{\frac{5}{3}}$$

points

$$(0,0,\sqrt{5}),$$

$$\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right)$$

$$\left(-\sqrt{\frac{10}{3}}, -\frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right)$$

$$f(0,0,\sqrt{5}) = 0$$

$$f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$

$$f\left(-\sqrt{\frac{10}{3}}, -\frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$