

- How to represent certain types of functions as sums of power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

- You might wonder why we would ever want to express a known function as a sum of infinitely many terms.

➤ Integration. (Easy to integrate polynomials) $\int e^{x^2} dx$

➤ Finding limit $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$

➤ Finding a sum of a series (not only geometric, telescoping) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!}$

Example: $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Example: Find Maclaurin series

$$f(x) = \cos x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

MEMORIZE: these Maclaurin Series

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Example:

Find Maclaurin series

$$f(x) = \tan^{-1} x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

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$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

The Maclaurin series for $f(x) = e^{-x^2/3}$ is given by

$$(a) \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{3^n \cdot n!}$$

$$(b) \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{3 \cdot n!}$$

$$(c) \sum_{n=1}^{+\infty} (-1)^n \frac{x^n}{3^n \cdot n!}$$

$$(d) \sum_{n=0}^{+\infty} \frac{x^{2n}}{3^n \cdot n!}$$

$$(e) \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n}}{9^n \cdot n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

The coefficient of x^{10} in the Maclaurin series of $f(x) = \sin(x^2)$ is equal to

(a) $\frac{1}{6}$

(b) 0

(c) $-\frac{1}{6}$

(d) $\frac{1}{120}$

(e) $\frac{1}{10}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

The Maclaurin series for $f(x) = x^2 \cos(\sqrt{2}x)$ is

(a)
$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^{2n+2}$$

(b)
$$\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{2})^n}{(2n)!} x^{2n}$$

(c)
$$\sum_{n=0}^{\infty} \frac{(\sqrt{2})^n}{(2n+1)!} x^{2n+1}$$

(d)
$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{2n} x^{2n+1}$$

(e)
$$\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{2}x)^{2n+2}}{(2n)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The coefficient of x^4 in the Maclaurin series of $\cos^2 x$ is

(a) $\frac{1}{4}$

(b) $\frac{2}{3}$

(c) 2

(d) $\frac{1}{2}$

(e) $\frac{1}{3}$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Example:

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

DEF:
$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Example:

$$\binom{1/3}{3} = \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} = \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6} = \frac{5}{81}$$

Example:

$$\binom{1/2}{5} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)(\frac{1}{2}-4)}{5!}$$

binomial series.

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

$$\binom{k}{n} = \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!}$$

NOTE:

$$\binom{k}{0} = 1 \quad \binom{k}{1} = \frac{k}{1!} = k \quad \binom{k}{2} = \frac{k}{2!} = \frac{k(k-1)}{2!}$$

The Binomial Series

Using the binomial series, we get $\sqrt[3]{1+x} =$
(for $|x| < 1$)

(a) $1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \dots$

(b) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \dots$

(c) $1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$

(d) $1 - \frac{1}{3}x + \frac{1}{6}x^2 + \frac{3}{27}x^3 + \dots$

(e) $1 + \frac{1}{3}x + \frac{1}{9}x^2 - \frac{5}{81}x^3 + \dots$

binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \quad R=1$$

If the Maclaurin series of $(1 + x)^{3/2}$ is

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots,$$

then $D + E =$

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(a) $-\frac{5}{128}$

(b) $\frac{9}{128}$

(c) $\frac{7}{16}$

(d) $-\frac{7}{16}$

(e) $-\frac{7}{128}$

binomial series.

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \quad R = 1$$

Important Maclaurin Series and Their Radii of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad R = \infty$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

Example: Find Maclaurin series

$$f(x) = \ln(1-x)$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Taylor series (center is a)

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots \end{aligned}$$

The first three terms of the Taylor series of $f(x) = \cos(2x)$ about $a = \pi$ are given by

(a) $1 - 2(x - \pi)^2 + \frac{2}{3}(x - \pi)^4$

(b) $1 - 2(x - \pi) - 2(x - \pi)^2$

(c) $1 - 2(x - \pi)^2 + 16(x - \pi)^4$

(d) $-1 + 2(x - \pi) + \frac{4}{3}(x - \pi)^3$

(e) $1 + 2(x + \pi)^2 - \frac{2}{3}(x + \pi)^4$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots \end{aligned}$$

The first three nonzero terms of the Taylor series of

$f(x) = \sin(2x)$ about $a = \frac{\pi}{2}$ are given by

(a) $-2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{2}\right)^5$

(b) $1 - 2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3$

(c) $-2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^2 - \frac{4}{15}\left(x - \frac{\pi}{2}\right)^3$

(d) $2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{2}\right)^5$

(e) $-2\left(x - \frac{\pi}{2}\right) + 4\left(x - \frac{\pi}{2}\right)^2 - 4\left(x - \frac{\pi}{2}\right)^5$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots \end{aligned}$$

The Taylor series of $f(x) = \frac{1}{x}$ about $x = 2$ is

(a)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x - 2)^{n+1}$$

(b)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x - 2)^n$$

(c)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x - 2)^n$$

(d)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x + 2)^n$$

(e)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x + 2)^n$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots \end{aligned}$$

Taylor series (center is a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

$$= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots$$

DEF:

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots$$

$$+ \frac{f^{(k)}(a)}{k!} (x - a)^k + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

Taylor polynomial of order n

The Taylor polynomial of order 3 generated by the function $f(x)=\ln(3+x)$ at $a=1$ is:

$$(a) \ln 4 + \frac{(x-1)}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192}$$

$$(b) \ln 4 - \frac{(x-1)}{4} + \frac{(x-1)^2}{32} - \frac{(x-1)^3}{192}$$

$$(c) \frac{(x-1)}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192} - \frac{(x-1)^4}{256}$$

$$(d) -\frac{(x-1)}{4} + \frac{(x-1)^2}{32} - \frac{(x-1)^3}{192} + \frac{(x-1)^4}{256}$$

$$(e) \ln 4 + \frac{(x-1)}{4} + \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192}$$

DEF:
$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$+ \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor polynomial of order n

The first three terms of the Taylor series of $f(x) = \frac{1}{\sqrt{x}}$ about $a = 4$ are given by

(a) $\frac{1}{4} - \frac{1}{16}(x - 4) + \frac{3}{4}(x - 4)^2$

(b) $\frac{1}{2} - \frac{1}{2}(x - 4) + \frac{3}{4}(x - 4)^2$

(c) $\frac{1}{2} - (x - 4) + (x - 4)^2$

(d) $\frac{1}{2} + \frac{1}{16}(x + 4) - \frac{1}{128}(x + 4)^2$

(e) $\frac{1}{2} - \frac{1}{16}(x - 4) + \frac{3}{256}(x - 4)^2$

The first four terms of the Taylor series of $f(x) = 4 + \ln x$ about $a = 1$ are given by

(a) $4 + (x - 1) - (x - 1)^2 + 2(x - 1)^3$

(b) $4 + (x + 1) - (x + 1)^2 + 2(x + 1)^3$

(c) $4 + (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

(d) $4 + 5(x - 1) - \frac{3}{2}(x - 1)^2 + (x - 1)^3$

(e) $4 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Taylor series (center is a)

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Taylor polynomial of order n

$$R_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Remainder

$$f(x) = P_n(x) + R_n(x)$$

Taylor Series

Taylor's Formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^k$$

Remainder consist of infinite terms

for some c between a and x.

REMARK: Observe that : $f^{(n+1)}(c)$ not $f^{(n+1)}(a)$

Taylor's Formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^k$$

for some c between a and x .

Taylor's Formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^k$$

for some c between 0 and x .

Taylor series (center is a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

DEF:

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$

$$= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

***n*th-degree Taylor polynomial of *f* at *a*.**

DEF:

$$R_n(x) = f(x) - T_n(x)$$

Remainder

Example:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$T_3(x) = \sum_{n=0}^3 x^n = 1 + x + x^2 + x^3$$

$$R_3(x) = \sum_{n=4}^{\infty} x^n = x^4 + x^5 + x^6 + \dots$$