JACOBIAN

The Jacobian, which turns out to be a 2 x 2 determinant, is a natural outgrowth of the changing of variables in an integral problem.

In a single integral you can change variables by letting x = g(u), dx = g'(u)du and obtain

$$\int_a^b f(x)dx = \int_c^d g(u)g'(u)du \quad with \;\; a = g(c), and \;\; b = g(d)$$

It's the double integral where we see the Jacobian.

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$$\iint_R f(x, y)\,dA = \iint_S f(g(u,v), h(u,v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

↑

This is the Jacobian

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

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Let's work out the Jacobian where we do the conversion of rectangular to polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$Hence,$$

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$