RANK OF A MATRIX

Let A be any m n matrix. Then A consists of n-column vectors a_1, $a_{2\ldots}$,....,a, which are m-vectors.

DEFINTION:

The rank of A is the maximal number of linearly independent column vectors in A, i.e. the maximal number of linearly independent vectors among {a_1, a_2,......, a}.

If A = 0, then the rank of A is 0.

We write r(A) or rk(A) for the rank of A. Note that we may compute the rank of any matrix-square or not.

RANK OF 2×2 MATRIX

Let us see how to compute 2×2 matrix:
EXAMPLE :

The rank of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

- $r(A)=2$ *if* $det(A)=ad-bc\neq 0$, since both column vectors are independent in this case.
- $r(A)=1$ *if* $det(A)=0$ but $A\neq 0=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since both column vectors are not linearly independent, but there is a single column vector that is linearly independent (i.e. non-zero).
- $r(A)=0$ *if* $A=0$

How do we compute $r(A)$ of m x n matrix?

COMPUTING RANK BY VARIOUS METHODS

1. BY GAUSS ELIMINATION
2. BY DETERMINANTS
3. BY MINORS
4. BY NORMAL FORM

1. USING GAUSS ELIMINATION

GAUSS ELIMINATION:

Use elementary row operations to reduce A to echelon form. The rank of A is the number of pivots or leading coefficients in the echelon form. In fact, the pivot columns (i.e. the columns with pivots in them) are linearly independent.

Note that it is not necessary to and the reduced echelon form – any echelon form will do since only the pivots matter.

POSSIBLE RANKS:

Counting possible number of pivots, we see that $rk(A) \leq m$ and $rk(A) \leq n$ for any m n matrix A.

EXAMPLE

Gauss elimination:

* Find the rank of a matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

SOLUTION:

We use elementary row operations:

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Since the echelon form has pivots in the first three columns,

A has rank, rk(A) = 3. The first three columns of A are linearly independent.

2. USING DETERMINANTS

Definition:

Let A be an m n matrix. A minor of A of order k is a determinant of a k X k sub-matrix of A.

We obtain the minors of order k from A by first deleting m rows and n -k columns, and then computing the determinant. There are usually many minors of A of a give order.

Example:

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

EXAMPLE

- **<u>COMPUTING MINORS:</u>**

We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$A = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1(-4) + 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4 x 4 sub-matrices of A.
There are 3.6 = 18 minors of order 2 and 3.4 = 12 minors of order 1.

3. USING MINORS

Proposition:

Let A be an m X n matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof:

If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank:

Start with the minors of maximal order k. If there is one that is non-zero, then rk(A) = k. If all maximal minors are zero, then rk(A) < k, and we continue with the minors of order k-1 and so on, until we and a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then rk(A) = 0.

RANK:EXAMPLES USING MINOR

Find the rank of the matrix

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution:

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\mathrm{rk}(A) = 3$.

- **EXAMPLE 2:**

Find the rank of the matrix

$$A= \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

Solution:

The maximal minors have order 3, so we compute the 4 minors of order 3.

The first one is

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7.(-1) + (-1).(-7) = 0$$

The other three are also zero. Since all minors of order 3 are zero, the rank must be $rk(A) < 3$. We continue to look at the minors of order two.

The first one is

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 5 - 18 = -13 \neq 0$$

It is not necessary to compute any more minors, and we conclude that $rk(A) = 2$. In fact, the rst two columns of A are linearly independent.

4. USING NORMAL FORM

NORMAL: A complex square matrix A is **normal** if

$$A^*A=AA^*$$

where A^* is the conjugate transpose of A. That is, a matrix is normal if it commutes with its conjugate transpose.

A matrix A with real entries satisfies $A^*=A^T$, and is therefore normal if $A^TA = AA^T$.

EXAMPLE:

Find the rank of a matrix using normal form,

$$A=\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Solution:

Reduce the matrix to echelon form,

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix} ---> \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now,convert the reduced matrix to normal form by using row/column operations,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} --> \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$

The given matrix is normal.

Now the rank will be defined by the suffix of the identity matrix ie 2.

$\therefore \text{rank}(A)=2$