Cayley-Hamilton theorem

Statement: A square matrix satisfies its own characteristic equation.

We saw that the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$.

$$\mathbf{M}^2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$$

and $\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1-5+6 & 5-5 \\ -10+10 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

This illustrates Cayley Hamilton Theorem.

Use the Cayley-Hamilton theorem to find $\mathbf{M}^6$ if $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$

$\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = 0$

$$
\begin{aligned}
\Rightarrow \quad & \mathbf{M}^2 = 5\mathbf{M} - 6\mathbf{I} \\
\Rightarrow \quad & \mathbf{M}^4 = (5\mathbf{M} - 6\mathbf{I})^2 \\
& = 25\mathbf{M}^2 - 60\mathbf{M} + 36\mathbf{I} \\
& = 25(5\mathbf{M} - 6\mathbf{I}) - 60\mathbf{M} + 36\mathbf{I} \\
& = 65\mathbf{M} - 114\mathbf{I} \\
& \mathrm{M}^6 = \mathrm{M}^4 \times \mathrm{M}^2 \\
& = (65\mathbf{M} - 114\mathbf{I})(5\mathbf{M} - 6\mathbf{I}) \\
& = 325\mathbf{M}^2 - 960\mathbf{M} + 684\mathbf{I} \\
& = 325(5\mathbf{M} - 6\mathbf{I}) - 960\mathbf{M} + 684\mathbf{I} \\
& = 665\mathbf{M} - 1266\mathbf{I} \\
& = 665\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 1266\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}
\end{aligned}
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