Complex and Unitary Matrices

Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix A, denoted by A^*, is given by

$$A^* = \bar{A}^T$$

where the entries of $\bar{A}$ are the complex conjugates of the corresponding entries of A.

Note that if A is a matrix with real entries, then $A^* = A^T$. To find the conjugate transpose of a matrix, first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

EXAMPLE 1 *Finding the Conjugate Transpose of a Complex Matrix*

Determine A^* for the matrix

$$A = \begin{bmatrix} 3+7i & 0 \\ 2i & 4-i \end{bmatrix}.$$

Solution

$$\overline{A} = \begin{bmatrix} \overline{3+7i} & \overline{0} \\ \overline{2i} & \overline{4-i} \end{bmatrix} = \begin{bmatrix} 3-7i & 0 \\ -2i & 4+i \end{bmatrix}$$

$$A^* = \overline{A}^T = \begin{bmatrix} 3-7i & -2i \\ 0 & 4+i \end{bmatrix}$$

Properties

If A and B are complex matrices and k is a complex number, then the following properties are true.

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(kA)^* = \bar{k}A^*$
4. $(AB)^* = B^*A^*$

Definition of a Unitary Matrix

A complex matrix A is **unitary** if

$$A^{-1} = A^*.$$

EXAMPLE 2 ***A Unitary Matrix***

Show that the matrix is unitary.

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+1 \end{bmatrix}$$

Solution Because

$$AA^* = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I_2,$$

you can conclude that $A^* = A^{-1}$. So, A is a unitary matrix.