Diagonalise Matrix

The goal here is to develop a useful factorization $A = PDP^{-1}$, when A is $n \times n$. We can use this to compute A^k quickly for large k.

The matrix D is a *diagonal* matrix (i.e. entries off the main diagonal are all zeros).

D^k is trivial to compute as the following example illustrates.

EXAMPLE: Let $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$. Compute D^2 and D^3. In general, what is D^k, where k is a positive integer?

Solution:

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 5^2 & 0 \\ 0 & 4^2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

and in general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}$$

EXAMPLE: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 1. Find the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda) = 0.$$

Eigenvalues of A: $\lambda = 1$ and $\lambda = 2$.

Step 2. Find three linearly independent eigenvectors of A.

By solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$, for each value of λ, we obtain the following:

Basis for $\lambda = 1$: $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = 2$: $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3: Construct P from the vectors in step 2.

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: Construct D from the corresponding eigenvalues.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 5: Check your work by verifying that AP = PD

$$AP = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$