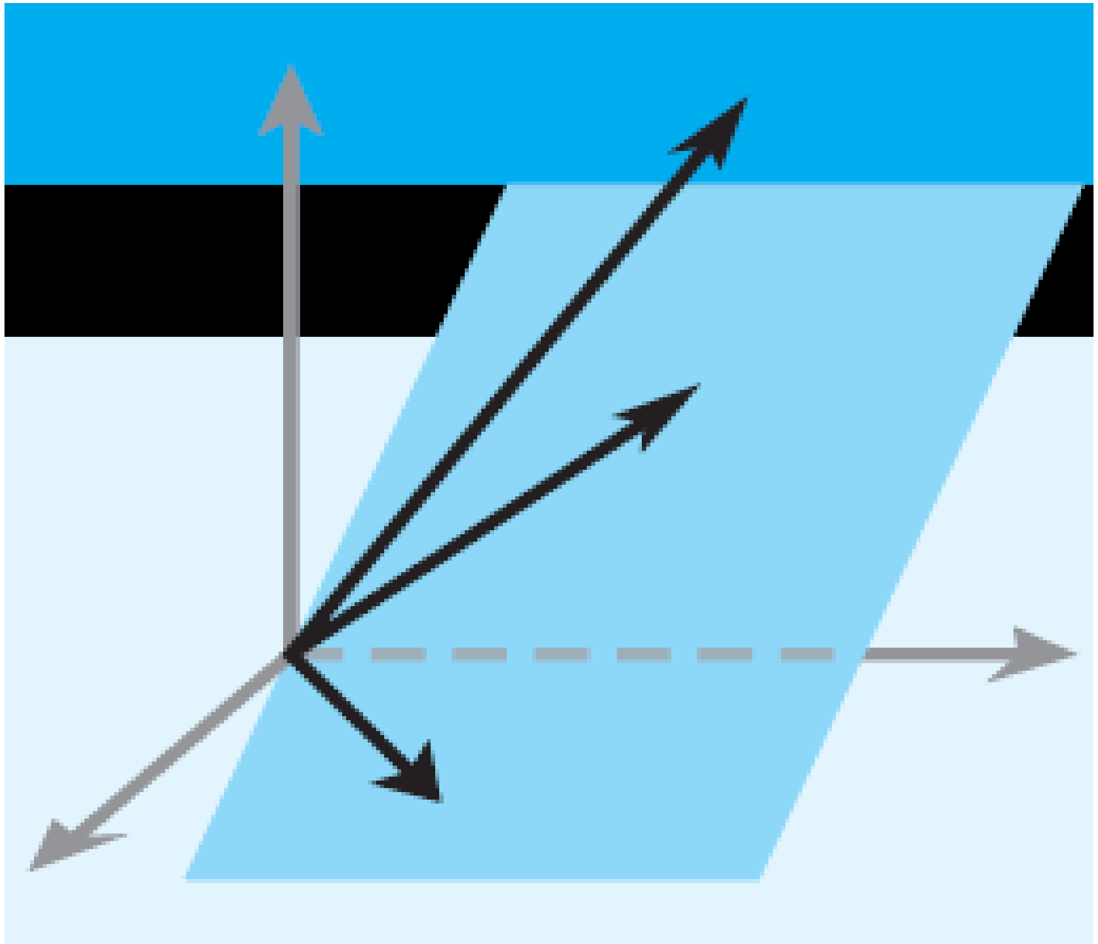
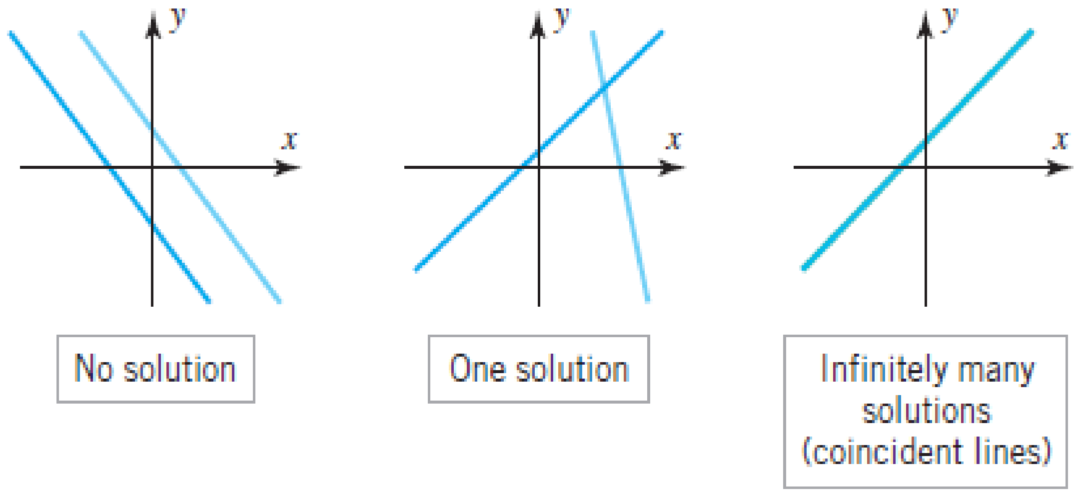
Linear Algebra

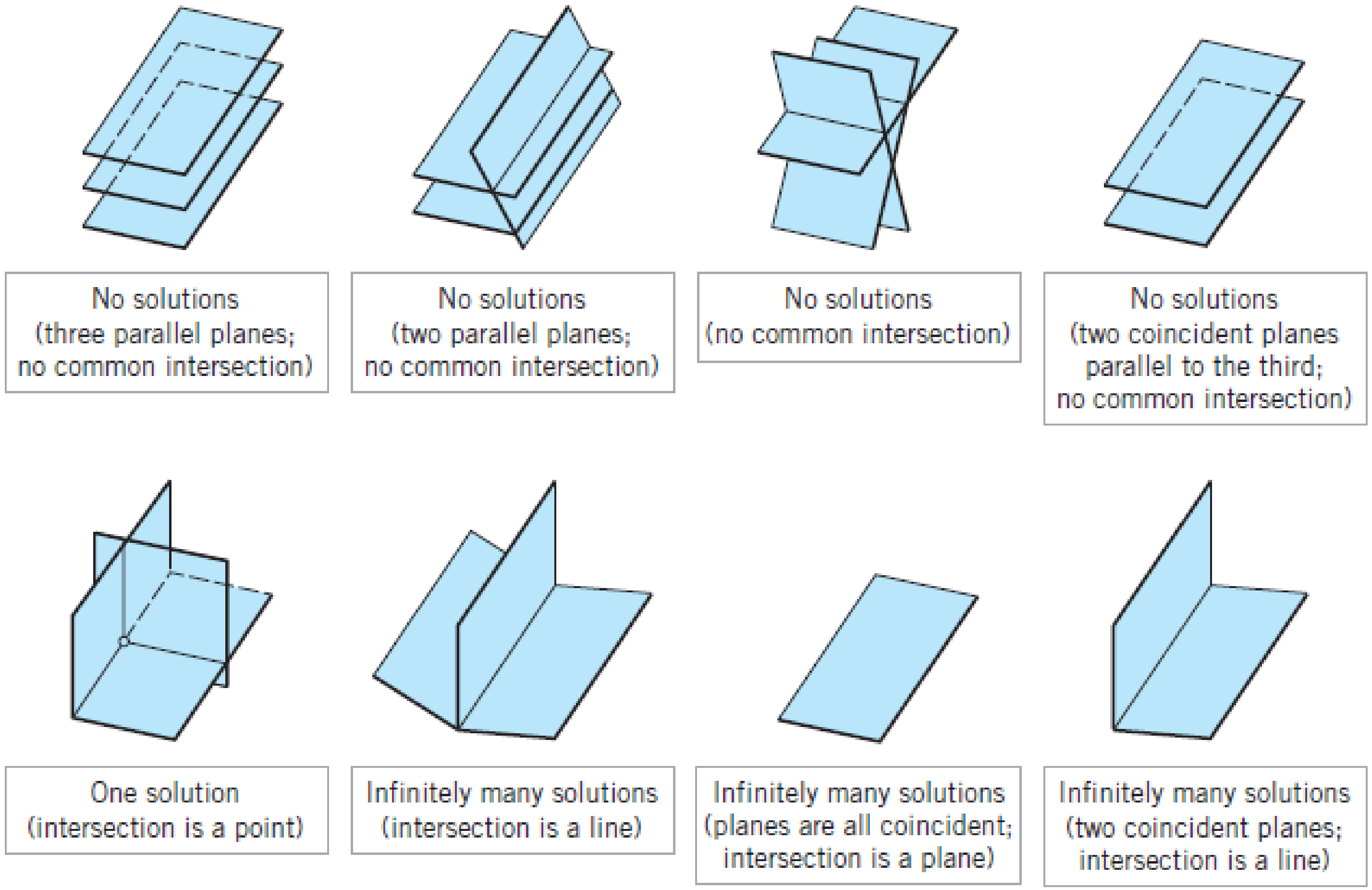


Linear Systems in Two Unknowns



▶ Figure 1.1.1

Linear Systems in Three Unknowns



▲ Figure 1.1.2

Elementary Row Operations

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another

Gaussian Elimination

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Reduced Row Echelon Form: Achieved by Gauss Jordan Elimination

Homogeneous Systems
All equations are set = 0

- **Theorem 1.2.1** If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n – r free variables
- **Theorem 1.2.2** A homogeneous linear system with more unknowns than equations has infinitely many solutions

Matrices and Matrix Operations

- **Definition 1** A matrix is a rectangular array of numbers. The numbers in the array are called the entries of the matrix.

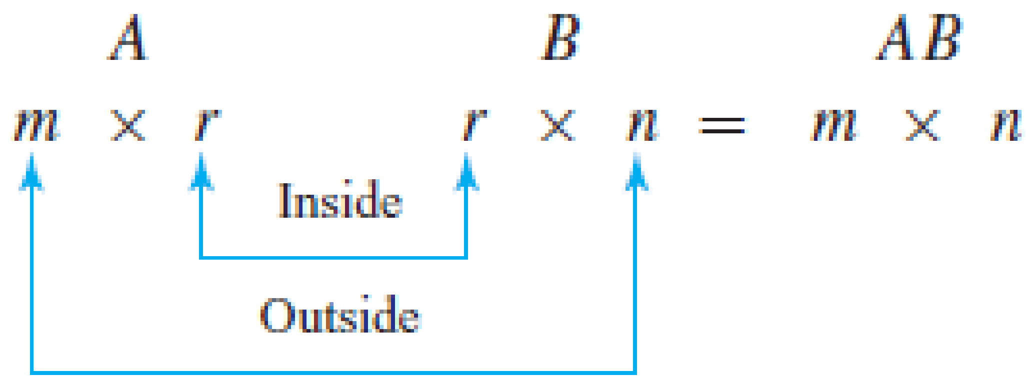
- **The size of a matrix M** is written in terms of the number of its rows x the number of its columns. A 2x3 matrix has 2 rows and 3 columns

Arithmetic of Matrices

- A + B: add the corresponding entries of A and B
- A – B: subtract the corresponding entries of B from those of A
- Matrices A and B must be of the same size to be added or subtracted
- cA (scalar multiplication): multiply each entry of A by the constant c

Multiplication of Matrices

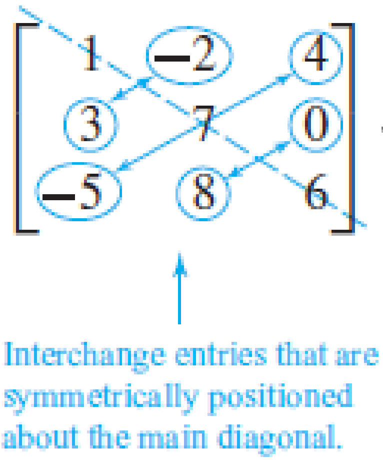
DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the *product* AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.



Transpose of a Matrix A^T

DEFINITION 7 If A is any $m \times n$ matrix, then the ***transpose of*** A, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow$$



$$\rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Transpose Matrix Properties

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Trace of a matrix

DEFINITION 8 If A is a square matrix, then the ***trace of*** A, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

EXAMPLE 11 Trace of a Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Algebraic Properties of Matrices

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **(Commutative law for addition)**

(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**

(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**

(d) $A(B + C) = AB + AC$ **(Left distributive law)**

(e) $(B + C)A = BA + CA$ **(Right distributive law)**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

The identity matrix and Inverse Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* (or *nonsingular*) and B is called an *inverse* of A. If no such matrix B can be found, then A is said to be *singular*.

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

Inverse of a 2x2 matrix

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

More on Invertible Matrices

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(b) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(c) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Using Row Operations to find A^{-1}

Begin with:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Use successive row operations to produce:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Linear Systems and Invertible Matrices

▶ **EXAMPLE 1 Solution of a Linear System Using A^{-1}**

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$. ◀

Diagonal, Triangular and Symmetric Matrices

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 lower triangular matrix

DEFINITION 1 A square matrix A is said to be *symmetric* if $A = A^T$.

RANK OF A MATRIX

Let A be any m n matrix. Then A consists of n-column vectors a_1, a_2...,....,a, which are m-vectors.

DEFINTION:

The rank of A is the maximal number of linearly independent column vectors in A, i.e. the maximal number of linearly independent vectors among {a_1, a_2,......, a}.

If A = 0, then the rank of A is 0.

We write r(A) or rk(A) for the rank of A. Note that we may compute the rank of any matrix-square or not.

RANK OF 2×2 MATRIX

Let us see how to compute 2×2 matrix:

EXAMPLE :

The rank of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

- $r(A)= 2$ *if* $det(A) = ad - bc \neq 0$, since both column vectors are independent in this case.
- $r(A)=1$ *if* $det(A)=0$ but $A \neq 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since both column vectors are not linearly independent, but there is a single column vector that is linearly independent (i.e. non-zero).
- $r(A)=0$ *if* $A=0$

How do we compute $r(A)$ of m x n matrix?

COMPUTING RANK BY VARIOUS METHODS

1. BY GAUSS ELIMINATION
2. BY DETERMINANTS
3. BY MINORS
4. BY NORMAL FORM

1. USING GAUSS ELIMINATION

GAUSS ELIMINATION:

Use elementary row operations to reduce A to echelon form. The rank of A is the number of pivots or leading coefficients in the echelon form. In fact, the pivot columns (i.e. the columns with pivots in them) are linearly independent.

Note that it is not necessary to and the reduced echelon form – any echelon form will do since only the pivots matter.

POSSIBLE RANKS:

Counting possible number of pivots, we see that $rk(A) \leq m$ and $rk(A) \leq n$ for any m n matrix A.

EXAMPLE

Gauss elimination:

* Find the rank of a matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

SOLUTION:

We use elementary row operations:

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Since the echelon form has pivots in the first three columns,

A has rank, rk(A) = 3. The first three columns of A are linearly independent.

2. USING DETERMINANTS

Definition:

Let A be an m n matrix. A minor of A of order k is a determinant of a k X k sub-matrix of A.

We obtain the minors of order k from A by first deleting m rows and n -k columns, and then computing the determinant. There are usually many minors of A of a give order.

Example:

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

EXAMPLE

- **<u>COMPUTING MINORS:</u>**

We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$A = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1(-4) + 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4 x 4 sub-matrices of A.
There are 3.6 = 18 minors of order 2 and 3.4 = 12 minors of order 1.

3. USING MINORS

Proposition:

Let A be an m X n matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof:

If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank:

Start with the minors of maximal order k. If there is one that is non-zero, then rk(A) = k. If all maximal minors are zero, then rk(A) < k, and we continue with the minors of order k-1 and so on, until we and a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then rk(A) = 0.

RANK:EXAMPLES USING MINOR

Find the rank of the matrix

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution:

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\mathrm{rk}(A) = 3$.

- **EXAMPLE 2:**

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

Solution:

The maximal minors have order 3, so we compute the 4 minors of order 3.

The first one is

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7.(-1) + (-1).(-7) = 0$$

The other three are also zero. Since all minors of order 3 are zero, the rank must be rk(A) < 3. We continue to look at the minors of order two. The first one is

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 5 - 18 = -13 \neq 0$$

It is not necessary to compute any more minors, and we conclude that rk(A) = 2. In fact, the rst two columns of A are linearly independent.

4. USING NORMAL FORM

<u>NORMAL:</u> A complex square matrix A is **normal** if

$$A^*A=AA^*$$

where A^* is the conjugate transpose of A. That is, a matrix is normal if it commutes with its conjugate transpose.

A matrix A with real entries satisfies $A^*=A^T$, and is therefore normal if $A^TA = AA^T$.

EXAMPLE:

Find the rank of a matrix using normal form,

$$A=\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Solution:

Reduce the matrix to echelon form,

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix} ---> \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now,convert the reduced matrix to normal form by using row/column operations,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} --> \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$

The given matrix is normal.

Now the rank will be defined by the suffix of the identity matrix ie 2.

$\therefore$ rank(A)=2