Multiple Integral

Double Integrals over Rectangles

Remark：

1. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $a = x_0 < x_1 < \cdots < x_n = b$

   Then P is called a partition of [a , b]

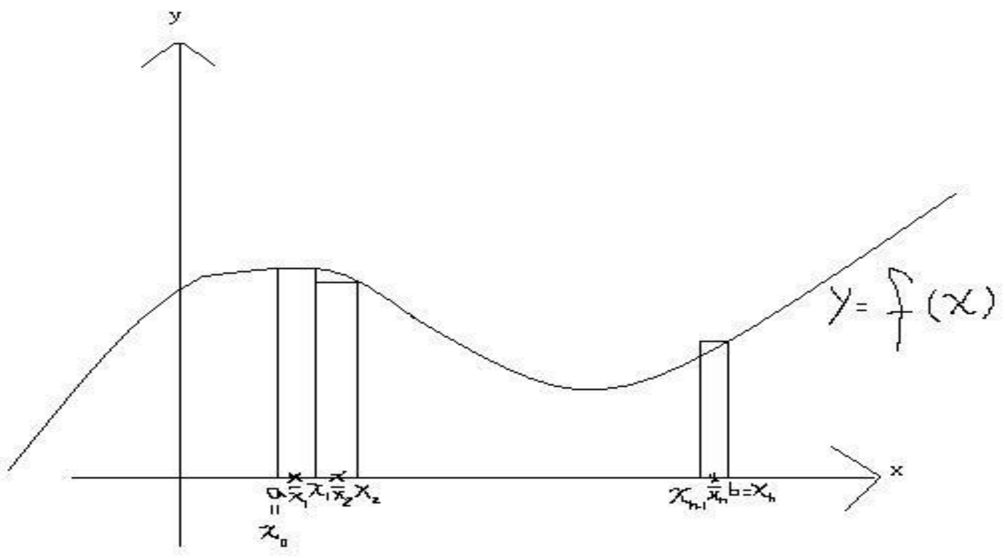
2. Define $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$

3. $|P| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ The norm of P

4. Choose $\bar{x}_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots n$, $\bar{x}_i$ is called a sample point.

5. The definite integral of f on [a , b]

$$\int_a^b f(x)dx = \lim_{|P|\to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \text{The area of } S$$



$$S = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

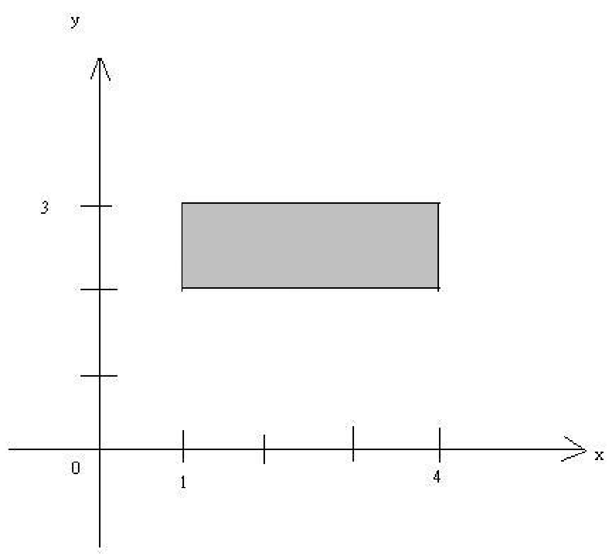
6. A closed rectangle R

$$R = [a, b] \times [c, d] = \{(x, y) \in R^2 \mid a \le x \le b, c \le y \le d\}$$

Example : Rectangle

$$1. [0,1] \times [2,4] = \{(x, y) \in R^2 \mid 0 \le x \le 1, 2 \le y \le 4\}$$

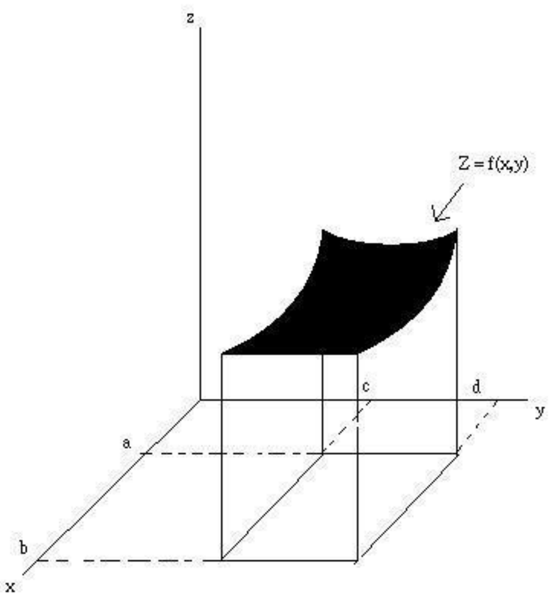
$$2. [1,2] \times [2,3]$$



Double Integrals and Volumes

Let $R = [a, b] \times [c, d]$, $S = \{(x, y, z) \in R^3 \mid (x, y) \in R, 0 \le z \le f(x, y)\}$

To find the volume of S - $V(S)$



Similarly to define $\int_a^b f(x)dx$

Divide the rectangle R into subintervals

$[a, b]$ is divided into m subinterval $[x_{i-1}, x_i]$

$i = 1, 2, \cdots m$, $\Delta x_i = x_i - x_{i-1}$, and $[c, d]$ is divided into n subinterval $[y_{j-1}, y_j]$, $\Delta y_j = y_j - y_{j-1}$, $j = 1, 2, \cdots n$

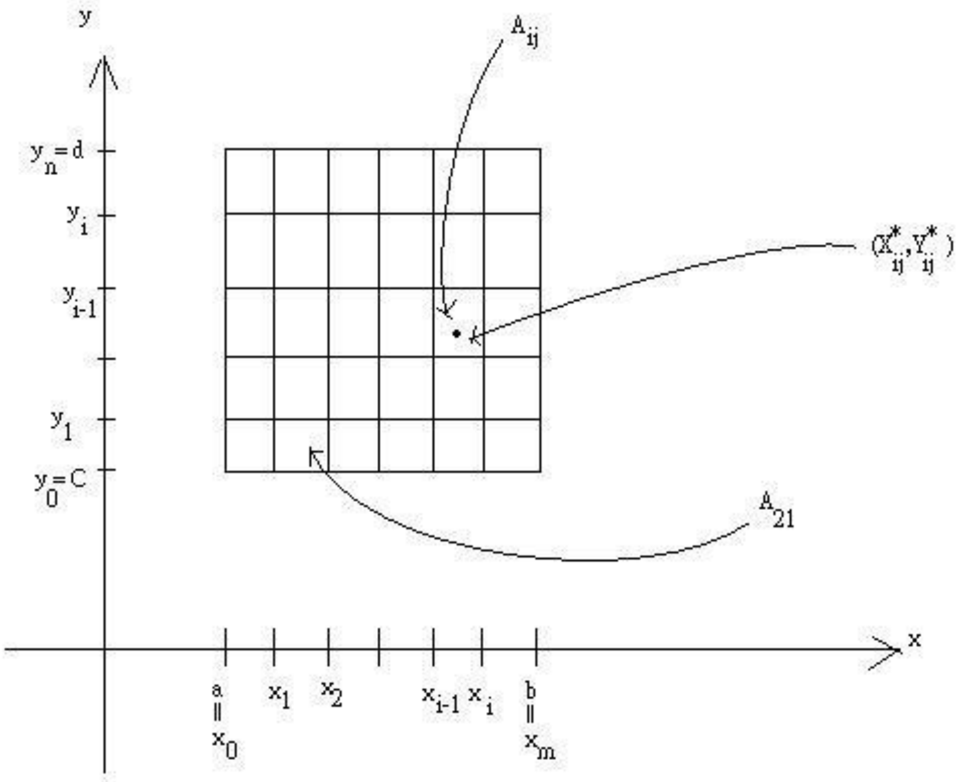
Define $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $i = 1, \cdots m$; $j = 1, \cdots n$

The area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$

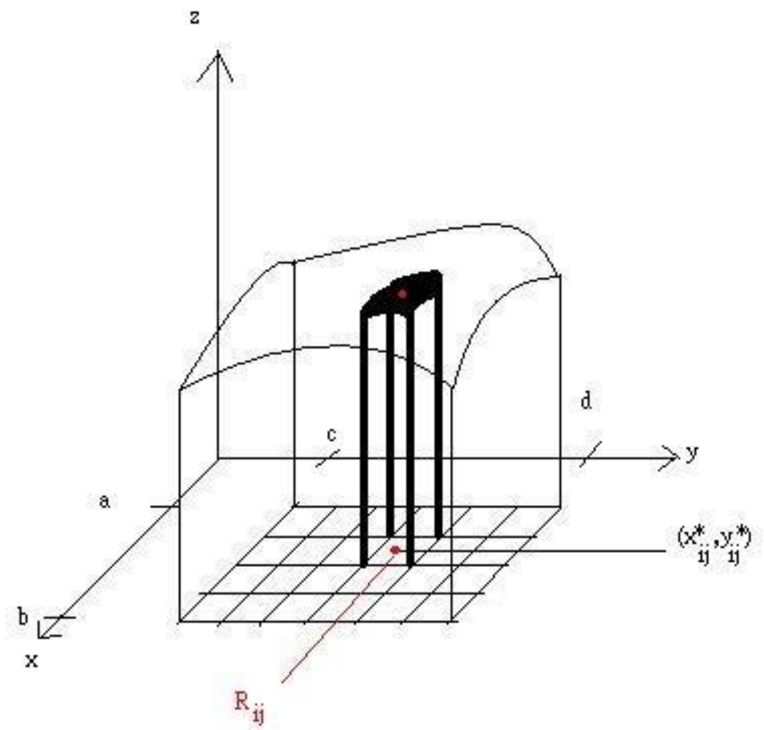
choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}

The volume of S can approximate $\sum_{j=1}^{n} \sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$

i.e $V(S) \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$



Let $|P|$ denote the length of the longest diagonal R_{ij}, $i = 1, 2 \cdots m$; $j = 1, 2 \cdots n$



Definition :

The double integral of f over the rectangle R is $\iint_R f(x, y)dA$

$$\iint_R f(x, y)dA = \lim_{|P|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\Delta A_{ij}$$

if this limit exists

properties:

1. $$\iint_R cf(x, y)dA = c\iint_R f(x, y)dA$$

2. $$\iint_R (f(x, y) + g(x, y))dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA$$

3. If $f(x, y) \geq g(x, y) \forall (x, y) \in R$, then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA$$

Definition :

1. f is integrable on R, if $\lim_{|P| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$ exist

2. $\iint_R f(x, y)\,dA$ is called the double integral of f over R

Theorem 1

Let f be bounded on the closed rectabgle R

(i) If f is continuous on R, then f is integrable on R

(ii) If f is continuous on R except on a finite number of smooth curves, then f is integrable on R

Definition:

1. If $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a, b)

2. If f is continuous at all $(a,b) \in R$, then f is continuous on R

Example:

1. $f(x,y) = \sin xy, (x,y) \in R = [0,\pi] \times [0,2\pi]$

   f is continuous on R

2. $f(x,y) = x^2 y + x, R = [0,\infty) \times [0,\infty)$

   f is continuous on R

3. $f(x,y) = \begin{cases} \dfrac{y}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

   f is not continuous on (0,1)

   f is not continuous on $(o, w), \forall w$

Iterated Integrals

Fixed y,Let $A(y) = \int_1^3 x^2 y\,dx = y \cdot \frac{x^3}{3}\Big|_1^3 = \frac{26}{3}y$

Consider $\int_0^2 A(y)dy = \int_0^2 (\int_1^3 x^2 y\,dx)dy$

Remark :

For function $f(x, y), R = [a, b] \times [c, d]$

1. Let $A(x) = \int_c^d f(x, y)dy, B(y) = \int_a^b f(x, y)dx$

$\int_a^b A(x)dx = \int_a^b (\int_c^d f(x, y)dy)dx$ is called an iterated integral

2. $\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx$

3. $\int_c^d \int_a^b f(x, y)\,dy\,dx = \int_c^d \left(\int_a^b f(x, y)\,dy \right) dx$

Example : evaluate

(i) $\int_0^3 \int_1^4 x^2 y\,dx\,dy$ (ii) $\int_0^3 \int_1^4 x^2 y\,dy\,dx$

(iii) $\int_0^4 \int_0^8 \frac{1}{4}(64 - 8x + y^2)\,dy\,dx$

(ιϖ) $\int_0^8 \int_0^4 \frac{1}{4}(64 - 8x + y^2)\,dx\,dy$

(ϖ) $\int_0^3 \int_1^2 (3x + 2y)\,dx\,dy$

Theorem 2 (Fubini's Theorem)

If f is continuous on $R = [a, b] \times [c, d]$, the

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Ex :

1. Find $\iint_R (x - 3y^2)\,dA$, where $R = [0,2] \times [1,3]$

2. Find $\iint_R y\sin xy\,dA$, where $R = [0,2] \times [0, \pi]$

Ans : $\iint_R y\sin xy\,dA = \int_0^2 \int_0^\pi y\sin xy\,dy\,dx = ?$

But $\int_0^\pi \int_0^2 y\sin xy\,dx\,dy = \int_0^\pi (-\cos xy)\Big|_0^2 dy = \int_0^\pi (1 - \cos 2y)\,dy$

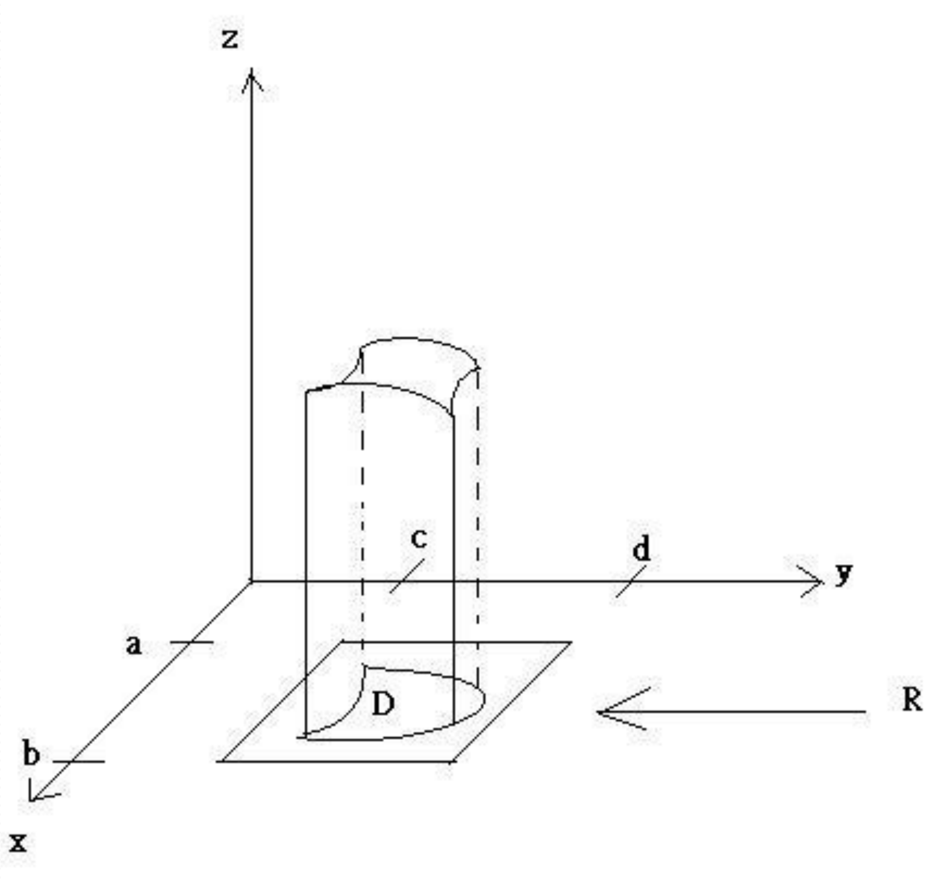
$$= \pi - (\frac{1}{2}\sin 2y\Big|_0^\pi) = \pi - 0 = \pi$$

3. Find $\iint_R x\cos xy\,dA$, where $R = [0, \pi] \times [0, \pi]$

4. Find $\iint_R \frac{1 + y}{1 + x}\,dA$, where $R = \{(x, y) \mid -1 \le x \le 2,\ 0 \le y \le 1\}$

Double Integral over General Regions

Let D be a bounded region and $D \subset R$, f is a function defined on D. Define a new function



$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \text{ is in R but not in D} \end{cases}$$

Definition :

1. The double integral of f over D is

$$\iint_D f(x, y)\,dA = \iint_R F(x, y)\,dA$$

2. A plane region D is said to be of typeI if

$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

where g_1, g_2 are twocontinuous function.

3. A plane region D is said to be of typeII if

$D = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$,

where h_1, h_2 are twocontinuous function.

Example :

1. $D_1 = \{(x, y) \mid 0 \le x \le \pi, \sin x \le y \le 1\}$, Type I

2. $D_2 = \{(x, y) \mid -1 \le y \le 1, 2y^2 \le x \le 1 + y^2\}$, Type II

Properties:

1. If f is continuous on a type I region D such that

$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

then $$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

2. If f is continuous on a type II region D then

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

where $D = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

Example : 1. Evaluate $\iint_D (x+3y)\,dA$

Where $D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$

Ans:

$$\iint_D (x+3y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+3y)\,dy\,dx$$

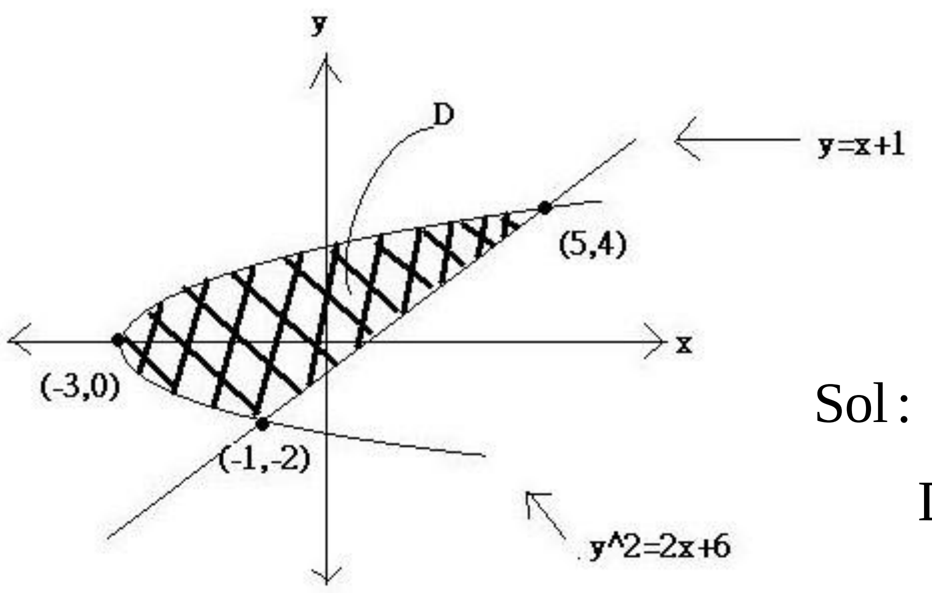
$$= \int_{-1}^{1} x(1+x^2-2x^2) + \frac{3}{2}((1+x^2)^2-(2x^2)^2)\,dx$$

$$= \int_{-1}^{1} x + x^3 - 2x^3 + \frac{3}{2} + 3x^2 + \frac{3}{2}x^4 - 4x^4\,dx$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{3}{2}x + x^3 - \frac{1}{2}x^5\right)\Bigg|_{-1}^{1} = \frac{3}{2} + 1 - \frac{1}{2} = 2$$

2. Evaluate $\iint_D xy\,dA$ where D is the region bounded by

the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



Sol:

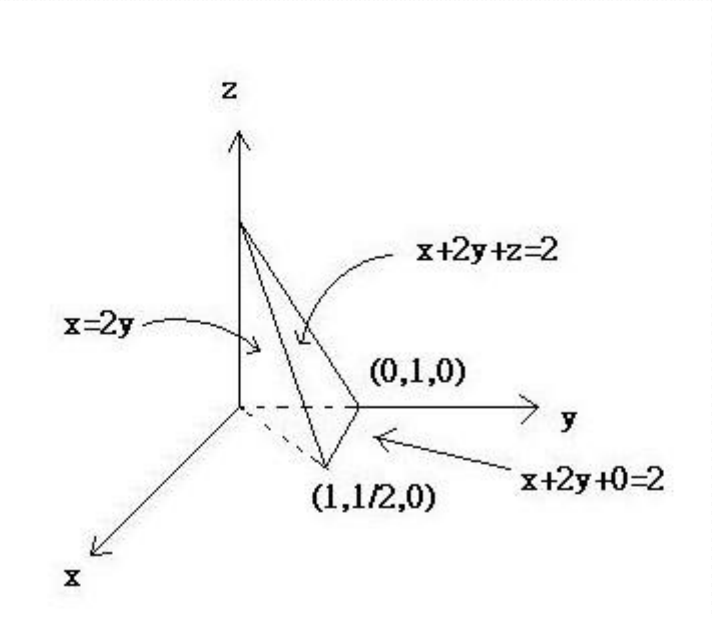
$$D = \{(x, y) \mid -3 \le x \le 5, ? \le y \le \sqrt{2x+6}\}$$

$$= \{(x, y) \mid \frac{y^2 - 6}{2} \le x \le y+1, -2 \le y \le 4\}$$

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = 36$$

3. Find the volume of the tetrahedron bounded by the planes $x = 2y, x = 0, z = 0$ and $x + 2y + z = 2$

Sol :



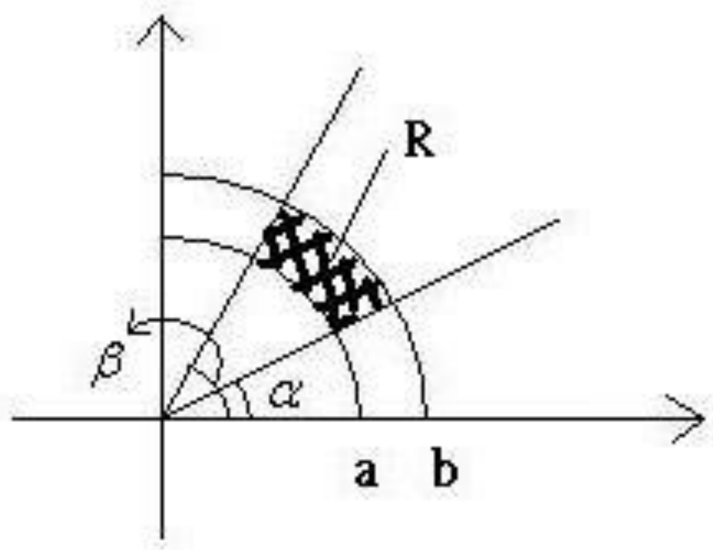
$$D = \{(x, y) | 0 \le x \le 1, \frac{x}{2} \le y \le \frac{2-x}{2}\}$$

$$所求\ V = \iint_D 2 - x - 2y dA = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2 - x - 2y) dy dx$$

$$= \frac{1}{3}$$

Double Integrals in Polar Coordinates

$$\text{Consider R} = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta$$



Polar rectangle

Example：

1. $R = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

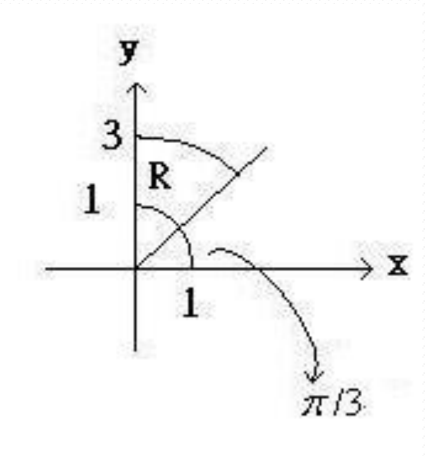
2. $R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}$

3. $R = \{(r,\theta) \mid 1 \le r \le 3, \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$

$$\text{The area of R is } A(R) = (\pi \cdot 3^2 - \pi \cdot 1^2) \frac{\frac{\pi}{2} - \frac{\pi}{3}}{2\pi}$$

$$= \frac{1}{2}(3^2 - 1^2) \cdot (\frac{\pi}{2} - \frac{\pi}{3})$$

$$= \frac{2}{3}\pi$$



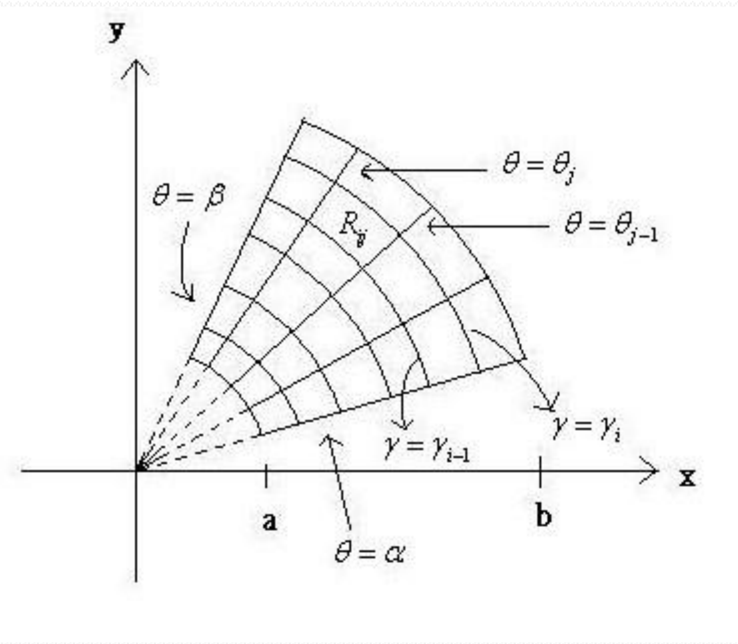
4. $R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$

The area of R_{ij} - ΔA_{ij} is

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j$$

$$= r_i^* \Delta r_i \Delta\theta_j$$

Where $\Delta r_i = r_i - r_{i-1},\ \Delta\theta_j = \theta_j - \theta_{j-1},\ i = 1,\cdots m;\ j = 1,\cdots n$



The Riemanu sum of f on R is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

$$\to \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Properties

1. Let $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ be a polar rectangle and $0 \le \beta - \alpha \le 2\pi$ If f is continuous on R, then

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

2. Let $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ be a polor region. If f is continuous on D then

$$\iint_D f(x, y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example :

1. Evaluate $\iint_R (4y^2 + 3x)dA$

where $R = \{(x, y) | y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$

Sol :

$$R = \{(x, y) | y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$
$$= \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

$$\iint_R (4y^2 + 3x)dA = \int_0^\pi \int_1^2 (4(r\sin\theta)^2 + 3r\cos\theta) r\,dr\,d\theta$$

$$= \int_0^\pi (15\sin^2\theta + 7\cos\theta)d\theta$$

$$= \frac{15}{2}\pi$$

2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$

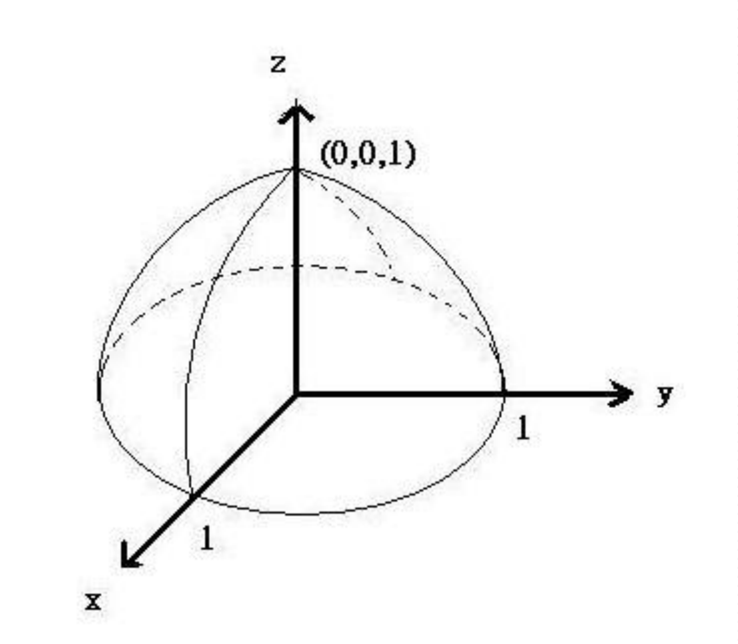
Sol:

$$D = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

$$V = \iint_D (1 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta$$

$$= \frac{\pi}{2}$$



Example :

Evaluate $\iint_{R^2} e^{-(x^2+y^2)} dA$

where $R^2 = \{(x, y) | -\infty < x < \infty, -\infty < y < \infty\}$

Sol :

Consider $D_n = \{(r, \theta) | 0 \le r \le n, 0 \le \theta \le 2\pi\}$

$$\iint_{R^2} e^{-(x^2+y^2)} dA = \lim_{n\to\infty} \iint_{D_n} e^{-(x^2+y^2)} dA$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \int_0^n e^{-r^2} r dr d\theta = \lim_{n\to\infty} \int_0^{2\pi} (\frac{1}{2} - \frac{1}{2} e^{-n^2}) d\theta$$

$$= \pi$$

The Cross Product

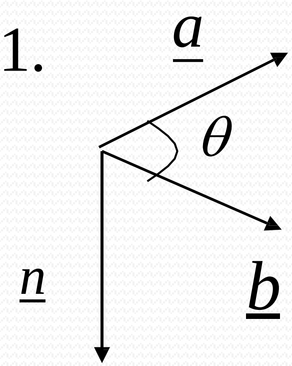
Definition

Let a, b be two nonzero three dimensional vectors

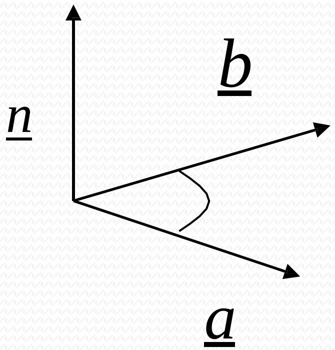
1. The inner product of $\underline{a}$ and $\underline{b}$ is $\underline{a} \cdot \underline{b} = |a||b|\cos\theta$
2. The cross product of $\underline{a}$ and $\underline{b}$ is the vector $\underline{a} \times \underline{b} = (|\underline{a}||\underline{b}|\sin\theta)\underline{n}$ where θ is the angle between $\underline{a}$ and $\underline{b}$, $0 \le \theta \le \pi$, and $\underline{n}$ is a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$ and whose direction is given by the right-hand rule: If the fingers of your right hand curl through the angle θ from $\underline{a}$ to $\underline{b}$, then your thumb points in the direction of $\underline{n}$

Example：

1.



2.



Properties

1. $\underline{a}$ and $\underline{b}$ are parallel if and only if $\underline{a} \times \underline{b} = \underline{0}$

2. $i = (1,0,0),\ j = (0,1,0),\ k = (0,0,1)$

(i) $i \times j = k,\ j \times i = -k$

(ii) $j \times k = i,\ k \times j = -i$

(iii) $k \times i = j,\ i \times k = -j$

3. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

4. Let c be a $_3$ a scalar

(i) $(c\underline{a})\times\underline{b} = c(\underline{a}\times\underline{b}) = \underline{a}\times(c\underline{b})$

(ii) $\underline{a}\times(\underline{b}+\underline{D}) = \underline{a}\times\underline{b}+\underline{a}\times\underline{D}$

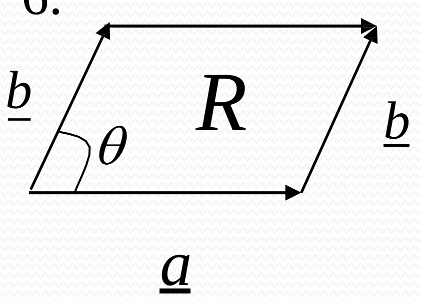
(iii) $(\underline{a}+\underline{b})\times\underline{D} = \underline{a}\times\underline{D}+\underline{b}\times\underline{D}$

5. If $\underline{a} = (a_1, a_2, a_3), \underline{b} = (b_1, b_2, b_3)$, then

$$a\times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

6.



The area of R is

$$A(R) = |\underline{a}\times\underline{b}|$$

Example

1. $\underline{a} = (1,2,0), \underline{b} = (2,-1,3)$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 6i - 3j - 5k$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 2 & 0 \end{vmatrix} = -6i + 3j + 5k$$

2. $\underline{a} = (1,3,4), \underline{b} = (2,7,-5)$

$\underline{a} \times \underline{b} = -43i + 13j + k$

3. $\underline{a} = i + j + k$, $\underline{b} = i + j - k$, Find $\underline{a} \times \underline{b}$

4. Find two unit vectors orthogonal to both $i + j$ and $i - j + k$

Triple Integrals

Rectangular box :

- $B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, e \le z \le f\}$

  $= [a, b] \times [c, d] \times [e, f]$

Example :

- 1. $B = [0,1] \times [1,3] \times [0,2]$

Definition :

- Let $B = [a, b] \times [c, d] \times [e, f]$ be a rectangular box. $[a, b]$ is divided into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, $[c, d]$ is a divided into m

- subintervals $[[y_{j-1}, y_j]$ of equal width Δy, $[e, f]$ is a divided into n subintervals $[[z_{k-1}, z_k]$ of equal width Δz

1. $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$

2. The volume of $B_{ijk} \rightarrow \Delta v = \Delta x \cdot \Delta y \cdot \Delta z$

3. The triple Riemann sum $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta v$

4. The tripleintergral of f over the box B is

$$\iiint_B f(x, y, z)\, dv = \lim_{l,m,n \rightarrow 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta v$$

if this limit exists

Theorem(Fubini's Theorem)

If f is continuous on $B = [a,b]\times[c,d]\times[e,f]$

then $\iiint_B f(x,y,z)dv = \int_e^f \int_c^d \int_a^b f(x,y,z)dxdydz$

- Example :
  - 1. Evaluate $\iiint_B xyz^2 dv$, where $B = [0,1]\times[1,2]\times[1,2]$
  - 2. Evaluate $\iiint_B (x + yz)dv$, where $B = [-1,1]\times[1,3]\times[0,2]$
  - 3. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 2, x \le z \le 1 - x - y\}$

    How to define $\iiint_E x^2 yz dv = ?$

For a general bounded region E. Consider a rectangular box

$$B \supset E, \text{ and define } F(x,y,z) = \begin{cases} f(x,y,z) & \text{if}(x,y,z) \in E \\ 0 & \text{if}(x,y,z) \notin E \end{cases}$$

$$\text{Define } \iiint_E f(x,y,z)dv = \iiint_B F(x,y,z)dv$$

properties

$$1.\ \text{If } E = \{(x,y,z) \mid (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$$

$$\text{then } \iiint_E f(x,y,z)dv = \iint_D \left(\int_{\phi_{1(x,y)}}^{\phi_{2(x,y)}} f(x,y,z)dz \right) dA$$

$$2.\ \text{If } E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), \phi_1(x,y) \le z \le \phi_2(x,y)\}$$

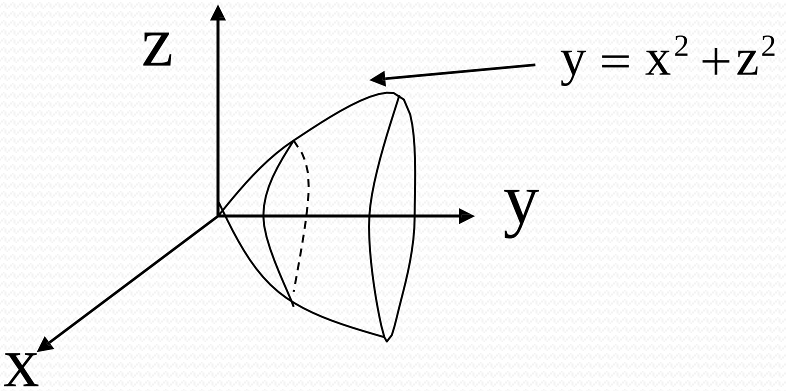
$$\text{then } \iiint_E f(x,y,z)dv = \int_a^b \int_{g_{1(x)}}^{g_{1(x)}} \int_{\phi_{1(x,y)}}^{\phi_{2(x,y)}} f(x,y,z)dzdydx$$

Examle :

1. $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}$

$$\iiint_E z\,dv = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

2. $E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}$



$$E = \{(x, y, z) \mid -2 \le x \le 2, -\sqrt{4 - x^2} \le z \le \sqrt{4 - x^2},\ z^2 + x^2 \le y \le 4\}$$

$$\iiint_E \sqrt{z^2 + x^2}\,dv = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z^2+x^2}^{4} \sqrt{z^2 + x^2}\,dy\,dz\,dx = \frac{128\pi}{15}$$