Change of Variables

Quite often, the evaluation of a double or triple integral is greatly simplified by a suit able change of variables:-

Let the variable x,y in the double integral $\iint_R f(x,y)dx\,dy$ be changed to u,v by means of relations

x=$x = \emptyset(u,v) = \varphi(u,v)$, then the double integral is transformed into $\iint_R f\{\emptyset(u,v),\varphi(u,v)\}|J|du\,dv$,

where $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Is the Jacobian of transformation from (x,y) to (u,v) co-ordinates and R' is the region in the uv-plane which corresponds to the region R in the xy-plane

(i) To change cartesian co-ordinates (x,y) to polar co-ordinates(r, θ)

Here we have x=r cosθ, y=r sin θ so that $x^2+y^2=r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r(cos^2\theta + sin^2\theta) = r$$

$$\iint_R f(x,y)dx\,dy = \iint_R f(r\,cos\theta, rsin\theta)r\,dr\,d\theta$$

$$i.e; replace\ x\ by\ r\ cos\theta, y\ by\ r\ sin\theta\ \ and\ dx\ dy\ r\ dr\ d\theta$$

(i) To change the Cartesian co-ordinates(x,y,z) to spherical polar co-ordinates (r, $\theta, \emptyset$)

Here we have

x=r sin θ $cos\emptyset$

y= r sin θ $cos\emptyset$

z=r cos θ

so that $x^2+y^2=r^2$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta cos\emptyset & rcos\theta cos\emptyset & -rsin\theta sin\emptyset \\ sin\theta sin\emptyset & rcos\theta sin\emptyset & rcos\emptyset\, sin\theta \\ cos\theta & -rsin\theta & 0 \end{vmatrix} = r^2 sin\theta$$

$$\iiint_V f(x,y,z)dx\,dy\,dz = \iiint_V f(rsin\theta\, cos\emptyset, rsin\theta\, sin\emptyset, rcos\theta) r^2 sin\theta\, dr\, d\theta\, d\emptyset$$

Note. Equation of the sphere $x^2+y^2+z^2=a^2$ in spherical polar co-ordinates is r=a

(i) If the region of integration is the whole sphere, then

$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \emptyset \le 2\pi$, then

(ii) If the region of the integration is the positive octant

$$0 \le r \le a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \emptyset \le \frac{\pi}{2}$$

(i) To change Cartesian co-ordinates (x,y,z) to cylindrical co-ordinates $(r, \emptyset, z)$

Here we have

$x = r\cos\emptyset$

$y = r\sin\emptyset$

$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \emptyset} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \emptyset} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \emptyset} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} cos\emptyset & -rsin\emptyset & 0 \\ sin\emptyset & rcos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

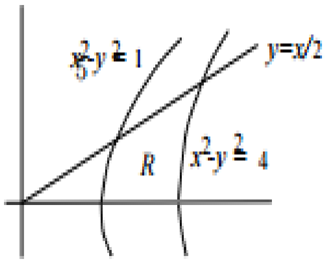
$$= r^2(cos^2\emptyset + sin^2\emptyset) = r$$

$$\iiint_V f(rcos\emptyset, rsin\emptyset, z)r\, dr\, d\emptyset\, dz$$

$Note - for\ the\ cylinder$ $x^2+y^2+z^2=a^2$, $z=0, z=h$, the limits of the integration are

$0 \le r \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \emptyset \le h$

If the region of integration is a cylinder (or cone), change the problem to cylindrical polar co-ordinates.



Example Evaluate $\iint_R \frac{y}{x}\,dx\,dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$, $y = x/2$.

Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x , and the integrand is y/x, this suggests making the change of variable

$$u = x^2 - y^2, \qquad v = \frac{y}{x} .$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} = 2 - 2y^2/x^2 = 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(1-v^2)} ,$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$\begin{aligned} \iint_R \frac{y}{x}\,dx\,dy &= \iint_R \frac{v}{2(1-v^2)}\,du\,dv \\ &= \int_0^{1/2} \int_1^4 \frac{v}{2(1-v^2)}\,du\,dv \\ &= -\frac{3}{4}\ln(1-v^2)\Big]_0^{1/2} = -\frac{3}{4}\ln\frac{3}{4} . \end{aligned}$$