Surface Area and Integrals

Surface Area

In the previous lecture we defined the surface area $a(S)$ of the parametric surface S, defined by $r(u,v)$ on T, by the double integral

$$a(S) = \iint_T \| r_u \times r_v \| \, dudv. \tag{1}$$

We will now drive a formula for the area of a surface defined by the graph of a function.

Area of a surface defined by a graph: Suppose a surface S is given by $z = f(x,y)$, $(x,y) \in T$, that is, S is the graph of the function $f(x,y)$. (For example, S is the unit hemisphere defined by $z = \sqrt{1 - x^2 - y^2}$ where (x,y) lies in the circular region $T : x^2 + y^2 \leq 1$.) Then S can be considered as a parametric surface defined by:

$$r(x,y) = xi + yj + f(x,y)k, \quad (x,y) \in T.$$

In this case the surface area becomes

$$a(S) = \iint_T \sqrt{1 + f_x^2 + f_y^2} \, dxdy. \tag{2}$$

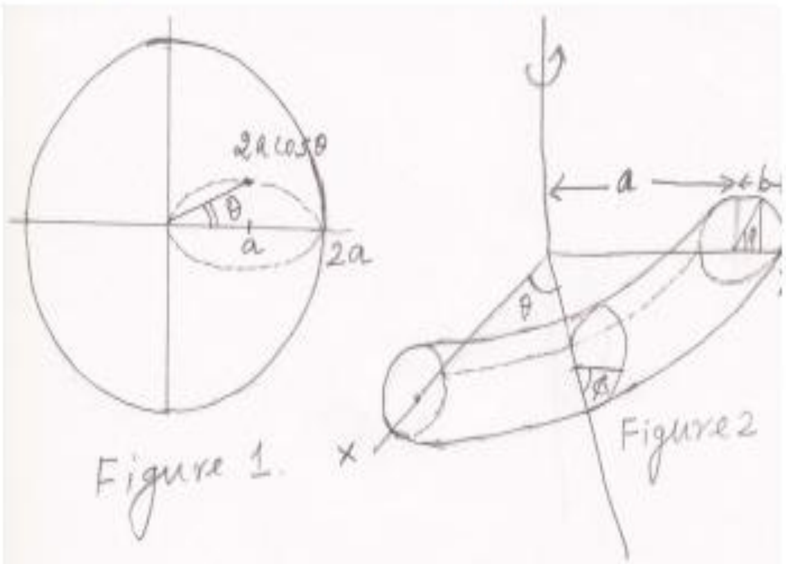
because $\| r_u \times r_v \| = \| -f_x i - f_y j + k \| = \sqrt{1 + f_x^2 + f_y^2}$.

Example 1: Let us find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$. Note that the sphere can be considered as a union of two graphs: $z = \pm\sqrt{4a^2 - x^2 - y^2}$. We will use the formula given in (2) to evaluate the surface area. Let $z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$. Then

$$f_x = \frac{-x}{\sqrt{4a^2-x^2-y^2}}, \quad f_y = \frac{-y}{\sqrt{4a^2-x^2-y^2}} \quad \text{and} \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{\frac{4a^2}{4a^2-x^2-y^2}}.$$

Let T be the projection of the surface $z = f(x, y)$ on the xy-plane (see Figure 1). Then, because of the symmetry, the surface area is

$$a(S) = 2\iint_T \sqrt{\frac{4a^2}{4a^2-x^2-y^2}}dxdy = 2 \times 2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \frac{2ardrd\theta}{\sqrt{4a^2-r^2}}.$$



Surface Integral

Surface Integrals: We will define the concept of integrals, called surface integrals, to the scalar functions defined on parametric surfaces. Surface integrals are used to define center of mass and moment of inertia of surfaces, and the surface integrals occur in several applications. We will not get in to the applications of the surface integrals in this course. We will define the surface integrals and see how to evaluate them.

Let S be a parametric surface defined by $r(u,v), (u,v) \in T$. Suppose r_u and r_v are continuous. Let $g : S \to \mathbb{R}$ be bounded. The surface integral of g over S, denoted by $\iint\limits_S g d\sigma$, is defined by

$$\iint\limits_S g \, d\sigma = \iint\limits_T g(r(u,v)) \parallel r_u \times r_v \parallel dudv = \iint\limits_T g(r(u,v))\sqrt{EG - F^2} \, dudv \qquad (4)$$

provided the RHS double integral exists. If S is defined by $z = f(x,y)$, then

$$\iint\limits_S g \, d\sigma = \iint\limits_T g[x,y,f(x,y)]\sqrt{1 + f_x^2 + f_y^2} \, dxdy. \qquad (5)$$

where T is the projection of the surface S over the xy-plane.

Example Let S be the hemispherical surface $z = (a^2 - x^2 - y^2)^{1/2}$. Let us evaluate $\iint_S \frac{d\sigma}{[x^2+y^2+(z+a)^2]^{1/2}}$.

We first parameterize the surface S as follows:

$$S := r(\theta, \phi) = (a \sin\phi \cos\theta,\ \ a \sin\phi \sin\theta,\ a \cos\phi),\ \ 0 \le \theta \le 2\pi,\ \ 0 \le \phi \le \pi.$$

Simple calculation shows that $\sqrt{EG - F^2} = a^2 \sin\phi$ and $[x^2+y^2+(z+a)^2]^{1/2} = 2a \cos\frac{\phi}{2}$. Therefore, by equation (4), the surface integral is

$$\iint_S \frac{d\sigma}{[x^2+y^2+(z+a)^2]^{1/2}} = \int_0^{2\pi} \int_0^{\pi/2} \frac{a^2 \sin\phi}{2a \cos\frac{\phi}{2}}\, d\phi d\theta.$$