Volume Cartesian and Polar Coordinates

Cartesian Coordinate

Example

Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

Solution: The volume of the rectangular integration region is

$$V = \int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx \quad \Rightarrow \quad V = 6.$$

The average of function f is:

$$\overline{f} = \frac{1}{6} \int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dy\, dx = \frac{1}{6} \Big[\int_0^1 x\, dx\Big] \Big[\int_0^2 y\, dy\Big] \Big[\int_0^3 z\, dz\Big]$$

$$\overline{f} = \frac{1}{6} \Big(\frac{x^2}{2}\Big|_0^1\Big) \Big(\frac{y^2}{2}\Big|_0^2\Big) \Big(\frac{z^2}{2}\Big|_0^3\Big) = \frac{1}{6} \Big(\frac{1}{2}\Big) \Big(\frac{4}{2}\Big) \Big(\frac{9}{2}\Big).$$

We conclude: $\overline{f} = 1/4$.

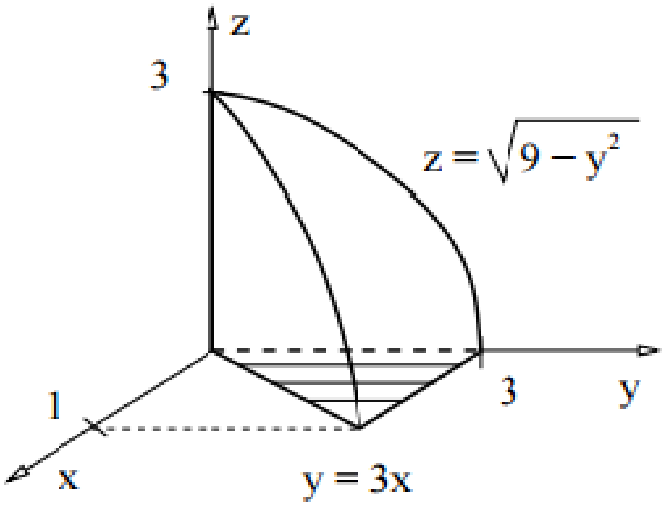
Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall:

$$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z\,dz\,dx\,dy.$$

For practice purpose only, let us change the integration order to $dz\,dy\,dx$:



The result is: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx.$

Polar Coordinates

EXAMPLE Find the volume under $z = \sqrt{4 - r^2}$ above the region enclosed by the curve $r = 2\cos\theta$, $-\pi/2 \le \theta \le \pi/2$; see figure 15.2.2. The region is described in polar coordinates by the inequalities $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$, so the double integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta.$$

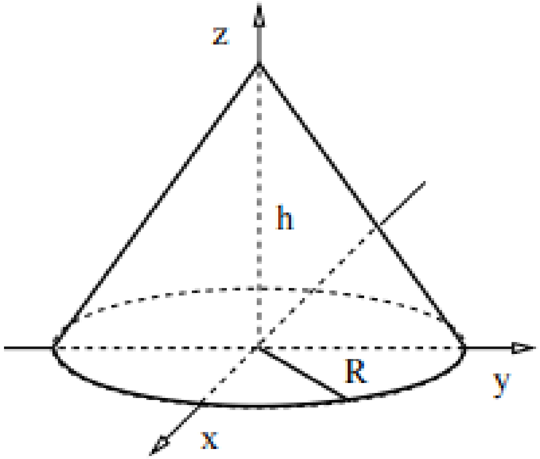
We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

$$\begin{aligned} 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta &= 2\int_0^{\pi/2} -\frac{1}{3}(4 - r^2)^{3/2}\Big|_0^{2\cos\theta} d\theta \\ &= 2\int_0^{\pi/2} -\frac{8}{3}\sin^3\theta + \frac{8}{3}\, d\theta \\ &= 2\left(-\frac{8}{3}\frac{\cos^3\theta}{3} - \cos\theta + \frac{8}{3}\theta\right)\Big|_0^{\pi/2} \\ &= \frac{8}{3}\pi - \frac{32}{9}. \end{aligned}$$

Example

Find the volume of a cone of base radius R and height h.

Solution: $R = \left\{ \theta \in [0, 2\pi],\ r \in [0, R],\ z \in \left[0, -\frac{h}{R} r + h\right] \right\}$.



$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^{R} \int_0^{h(1-r/R)} dz\,(r\,dr)\,d\theta, \\
&= h \int_0^{2\pi} \int_0^{R} \left(1 - \frac{r}{R}\right) r\,dr\,d\theta, \\
&= h \int_0^{2\pi} \int_0^{R} \left(r - \frac{r^2}{R}\right) dr\,d\theta, \\
&= h \left(\frac{R^2}{2} - \frac{R^3}{3R}\right) \int_0^{2\pi} d\theta = 2\pi h R^2 \frac{1}{6}.
\end{aligned}
$$

We conclude: $V = \frac{1}{3}\pi R^2 h$.