Special functions

Gamma and Beta Functions

The factorial function

$$\int_0^\infty e^{-ax}dx = -\frac{1}{\alpha}e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha} \quad (\alpha > 0)$$

$$\int_0^\infty xe^{-ax}dx = -\frac{1}{\alpha}xe^{-\alpha x}\Big|_0^\infty - \int_0^\infty\left(-\frac{1}{\alpha}e^{-\alpha x}\right)dx = \frac{1}{\alpha^2}.$$

Similarly, $\int_0^\infty x^2e^{-\alpha x}dx = \frac{2}{\alpha^3}$, $\int_0^\infty x^3e^{-\alpha x}dx = \frac{2\cdot 3}{\alpha^4}$

$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}} \rightarrow \int_0^\infty x^n e^{-x}dx = n! \ (\alpha = 1)$$

Definition of the gamma function: recursion relation

- Gamma function $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \quad p > 0.$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!,$$

$$\Gamma(n+1) = \int_0^\infty x^{n} e^{-x} dx = n!.$$

- Recursion relation $\Gamma(p+1) = \int_0^\infty x^{p} e^{-x} dx = p!, \quad p > -1.$

$$\Gamma(p+1) = p\Gamma(p)$$

- Example $\Gamma(9/4) = (5/4)\Gamma(5/4) = (5/4)(1/4)\Gamma(1/4)$

$$so, \ \Gamma(1/4) \div \Gamma(9/4) = 16/5.$$

The Gamma function of negative numbers

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1) \quad (p<0)$$

- Example

$$\Gamma(-0.3) = \frac{1}{-0.3}\Gamma(0.7), \quad \Gamma(-1.3) = \frac{1}{(-0.3)(-1.3)}\Gamma(0.7).$$

$$cf. \quad \Gamma(p) = \frac{1}{p}\Gamma(p+1) \to \infty \quad \text{as} \quad p \to 0.$$

- Using the above relation,
  1) Gamma(p= negative integers) $\to$ infinite.
  2) For $p < 0$, the sign changes alternatively in the intervals between negative integers

Some important formulas involving gamma functions

- $\Gamma(1/2) = \sqrt{\pi}$

(prove) $\Gamma(1/2) = \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt = \int_0^{\infty} \frac{1}{y} e^{-y^2} 2y dy = 2\int_0^{\infty} e^{-y^2} dy.$

$$[\Gamma(1/2)]^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx dy = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r dr d\theta = \pi.$$

- $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin \pi p}.$

Beta functions

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad p>0,\ q>0. \quad cf.\ B(p,q)=B(q,p)$$

$$i)\quad B(p,q)=\int_0^a\left(\frac{y}{a}\right)^{p-1}\left(1-\frac{y}{a}\right)^{q-1}\frac{dy}{a}=\frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}dy.\quad (x=y/a)$$

$$ii)\quad B(p,q)=2\int_0^{\pi/2}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta.\quad \left(x=\sin^2\theta\right)$$

$$iii)\quad B(p,q)=\int_0^\infty\frac{y^{p-1}dy}{(1+y)^{p+q}}.\quad (x=y/(1+y))$$

Beta functions in terms of gamma functions

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Prove)

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt=2\int_0^\infty y^{2p-1}e^{-y^2}dy,\quad \Gamma(q)=2\int_0^\infty x^{2q-1}e^{-x^2}dx$$

$$\Gamma(p)\Gamma(q)=4\int_0^\infty\int_0^\infty x^{2q-1}y^{2p-1}e^{-(x^2+y^2)}dxdy$$

$$=4\int_0^\infty\int_0^{\pi/2}(r\cos\theta)^{2q-1}(r\sin\theta)^{2p-1}e^{-r^2}r\,dr\,d\theta$$

$$=4\int_0^\infty r^{2p+2q-1}e^{-r^2}dr\int_0^{\pi/2}(\cos\theta)^{2q-1}(\sin\theta)^{2p-1}d\theta=\frac{1}{2}\Gamma(p+q)\cdot\frac{1}{2}B(p,q).$$

- Example $I = \int_0^\infty \frac{x^3 dx}{(1+x)^5}$ $cf.$ $B(p,q) = \int_0^\infty \frac{y^{p-1} dy}{(1+y)^{p+q}}.$

$$p + q = 5, \quad p - 1 = 3 \;\rightarrow\; p = 4, \;\; q = 1.$$

$$\frac{\Gamma(4)\Gamma(1)}{\Gamma(5)} = \frac{3!}{4!} = \frac{1}{4}.$$

LEGENDRE'S DUPLICATION FORMULA

$$\Gamma(1+z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\;\Gamma(2z+1)$$

General proof in §13.3.

Proof for $z = n =$ 1, 2, 3, …. :

(Case $z = 0$ is proved by inspection.)

$$\Gamma\left(n+\frac{1}{2}\right) = \left[\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\,\frac{1}{2}\right]\sqrt{\pi}$$

$$\Gamma(z) = (z-1)\,\Gamma(z-1)$$

$$= \frac{1}{2^n}\Big[\,(2n-1)(2n-3)\cdots\Big]\sqrt{\pi} \qquad = \frac{1}{2^n}(2n-1)!!\,\sqrt{\pi}$$

$$\Gamma(1+n) = n! \;= \frac{(2n)!!}{2^n}$$

$$\rightarrow$$

$$\Gamma(1+n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!!}{2^{2n}}(2n-1)!!\,\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}}\sqrt{\pi}$$

13.3. THE BETA FUNCTION

Beta Function :

$$B(p,q) \equiv \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(q,p)$$

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty ds\, e^{-s^2}\, s^{2p-1} \int_0^\infty dt\, e^{-t^2}\, t^{2q-1}$$

$$\Gamma(z) = 2\int_0^\infty ds\, e^{-s^2}\, s^{2z-1}$$

$$s = r\cos\theta \qquad t = r\sin\theta \quad \to \quad ds\,dt = \begin{vmatrix} \partial_r s & \partial_\theta s \\ \partial_r t & \partial_\theta t \end{vmatrix} dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty r\,dr \int_0^{\pi/2} d\theta\, e^{-r^2}\, r^{2(p+q-1)} \cos^{2p-1}\theta\,\sin^{2q-1}\theta$$

$$= 2\,\Gamma(p+q) \int_0^{\pi/2} d\theta\, \cos^{2p-1}\theta\,\sin^{2q-1}\theta$$

$$p = m+1 \qquad q = n+1$$

$$\to B(p,q) = 2\int_0^{\pi/2} d\theta\, \cos^{2p-1}\theta\,\sin^{2q-1}\theta$$

$$\frac{m!\,n!}{(m+n+1)!} = 2\int_0^{\pi/2} d\theta\, \cos^{2m+1}\theta\,\sin^{2n+1}\theta$$

ALTERNATE FORMS : DEFINITE INTEGRALS

$$B(p+1,q+1) = 2\int_0^{\pi/2} d\theta \;\cos^{2p+1}\theta\;\sin^{2q+1}\theta$$

$$t = \cos^2\theta \quad\rightarrow\quad B(p+1,q+1) = \int_0^1 d\cos^2\theta\;\cos^{2p}\theta\,\sin^{2q}\theta \;= \int_0^1 dt\;\; t^p\,(1-t)^q$$

$$t = x^2 \quad\rightarrow\quad B(p+1,q+1) = 2\int_0^1 dx\, x^{2p+1}\left(1-x^2\right)^q$$

$$t = \frac{u}{1+u} \quad\rightarrow\quad dt = \left(\frac{1}{1+u} - \frac{u}{(1+u)^2}\right)du \;= \frac{1}{(1+u)^2}\,du \qquad 1-t = \frac{1}{1+u}$$

$$B(p+1,q+1) = \int_0^{1/2} du\,\frac{1}{(1+u)^2}\left(\frac{u}{1+u}\right)^p\left(\frac{1}{1+u}\right)^q \;= \int_0^{1/2} du\,\frac{u^p}{(1+u)^{p+q2}}$$

To be used in integral rep. of Bessel (Ex.14.1.17)
& hypergeometric (Ex.18.5.12) functions

DERIVATION: LEGENDRE DUPLICATION FORMULA

$$B(p+1,q+1)=\int_0^1 dt\ t^p\,(1-t)^q\ =2\int_0^1 dx\, x^{2p+1}\,(1-x^2)^q$$

$$t=\frac{1}{2}(1+s)$$

$$\rightarrow\qquad B\left(z+\frac{1}{2},z+\frac{1}{2}\right)=\int_0^1 dt\ t^{z-1/2}\,(1-t)^{z-1/2}\ =\left(\frac{1}{2}\right)^{2z}\int_{-1}^1 ds\ (1+s)^{z-1/2}\,(1-s)^{z-1/2}$$

$$=\left(\frac{1}{2}\right)^{2z}\int_{-1}^1 ds\ (1-s^2)^{z-1/2}\ =\left(\frac{1}{2}\right)^{2z}2\int_0^1 ds\ (1-s^2)^{z-1/2}\ =\left(\frac{1}{2}\right)^{2z}B\left(\frac{1}{2},z+\frac{1}{2}\right)$$

$$\rightarrow\qquad \frac{\Gamma\left(z+\frac{1}{2}\right)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z+1)}=\left(\frac{1}{2}\right)^{2z}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(z+1)}$$

$$\Gamma(z+1)\Gamma\left(z+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2z}}\Gamma(2z+1)$$