

Dirichlet's Integral and Applications

DIRICHLET'S INTEGRAL:

If l, m, n are all positive, then the triple integral

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

Where V is the region $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$.

Note:

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} h^{l+m+n}$$

Where V is the domain, $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq h$

Corollary:

Dirichlet's theorem for n variables, the theorem states that

$$\iiint \dots \int x_1^{l_1-1} x_2^{l_2-1} \dots x_n^{l_n-1} dx_1 dx_2 dx_3 \dots dx_n = \frac{\Gamma l_1 \Gamma l_2 \Gamma l_3 \dots \Gamma l_n}{\Gamma(1 + l_1 + l_2 + \dots + l_n)} h^{l_1+l_2+\dots+l_n}$$

Liouville's extension of dirichlet theorem:

$$\begin{aligned} & \iiint f(x + y + z) x^{l-1} y^{m-1} z^{n-1} dx dy dz \\ &= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u) u^{l+m+n-1} du \end{aligned}$$

Example 1: Show that $\iiint \frac{dx dy dz}{(x+y+z+1)^3} = \frac{1}{2} \log 2 - \frac{5}{16}$,

the integral being taken throughout the volume bounded by $x = 0, y = 0, z = 0, x + y + z = 1$.

Solution: By Liouville's theorem, when $0 < x + y + z < 1$

$$\begin{aligned} \iiint \frac{dx dy dz}{(x+y+z+1)^3} &= \iiint \frac{x^{l-1} y^{m-1} z^{n-1} dx dy dz}{(x+y+z+1)^3} && (0 \leq x + y + z \leq 1) \\ &= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3} u^{3-1} du \\ &= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3} du \\ &= \int_0^1 \left[\frac{1}{u+1} - \frac{2}{(u+1)^2} + \frac{1}{(u+1)^3} \right] du && \text{(Partial fractions)} \\ &= \frac{1}{2} \left[\log(u+1) + \frac{2}{u+1} - \frac{1}{2(u+1)^2} \right]_0^1 \\ &= \frac{1}{2} \left[\log 2 + 2 \left(\frac{1}{2} - 1 \right) - \left(\frac{1}{8} - \frac{1}{2} \right) \right] \\ &= \frac{1}{2} \log 2 - \frac{5}{16} \\ \therefore \iiint \frac{dx dy dz}{(x+y+z+1)^3} &= \frac{1}{2} \log 2 - \frac{5}{16} \end{aligned}$$

Example 2: Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the density at any point being $\rho = k x y z$.

Solution:
$$\begin{aligned} \text{Mass} &= \iiint \rho \, dv = \iiint (k x y z) dx dy dz \\ &= k \iiint (x dx)(y dy)(z dz) \end{aligned} \quad \text{_____ (1)}$$

Putting $\frac{x^2}{a^2} = u, \frac{y^2}{b^2} = v, \frac{z^2}{c^2} = w$ and $u + v + w = 1$

So that $\frac{2x dx}{a^2} = du, \frac{2y dy}{b^2} = dv, \frac{2z dz}{c^2} = dw$

Mass = $k \iiint \left(\frac{a^2 du}{2}\right) \left(\frac{b^2 dv}{2}\right) \left(\frac{c^2 dw}{2}\right)$
 $= \frac{k a^2 b^2 c^2}{8} \iiint du dv dw, \quad \text{Where } u + v + w \leq 1$

$= \frac{k a^2 b^2 c^2}{8} \iiint u^{l-1} v^{l-1} w^{l-1} du dv dw$

$= \frac{k a^2 b^2 c^2}{8} \frac{\Gamma 1 \Gamma 1 \Gamma 1}{\Gamma 3+1} = \frac{k a^2 b^2 c^2}{8 \times 6}$

$= \frac{k a^2 b^2 c^2}{48}$

$\therefore \text{Mass} = \frac{k a^2 b^2 c^2}{48}$