Curl of a Vector

If $A = a_x \, i + a_y \, j + a_z \, k$, the curl of A is defined by

$$\text{curl } A = \nabla \times A$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \times (a_x \, i + a_y \, j + a_z \, k)$$

$$\Rightarrow \text{curl } A = \nabla \times A = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}.$$

Example :

If $A = (y^4 - x^2 z^2)\, i + (x^2 + y^2)\, j - x^2 yz\, k$,

determine $curl\, A$ at $(1,3,-2)$.

Solution

$$curl\ \underset{\sim}{A} = \nabla \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^4 - x^2 z^2 & x^2 + y^2 & -x^2 yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x^2 yz) - \frac{\partial}{\partial z}(x^2 + y^2) \right) \underset{\sim}{i}$$

$$- \left(\frac{\partial}{\partial x}(-x^2 yz) - \frac{\partial}{\partial z}(y^4 - x^2 z^2) \right) \underset{\sim}{j}$$

$$+ \left(\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2 z^2) \right) \underset{\sim}{k}$$

$$= -x^2 z\, \underset{\sim}{i} - (-2xyz + 2x^2 z)\, \underset{\sim}{j} + (2x - 4y^3)\, \underset{\sim}{k}.$$

At $(1,3,-2)$,

$$curl\ \underset{\sim}{A} = -(1)^2(-2)\underset{\sim}{i} - (-2(1)(3)(-2) + 2(1)^2(-2))\underset{\sim}{j}$$

$$+ (2(1) - 4(3)^3)\underset{\sim}{k}$$

$$= 2\underset{\sim}{i} - 8\underset{\sim}{j} - 106\underset{\sim}{k}.$$

Exercise:

If $\underset{\sim}{A} = (xy^3 - y^2z^2)\underset{\sim}{i} + (x^2 + z^2)\underset{\sim}{j} - x^2yz^2\underset{\sim}{k}$,

determine $curl\ \underset{\sim}{A}$ at point $(1,2,3)$.

Answer

$$curl\ \underset{\sim}{A} = (-x^2z^2 - 2z)\underset{\sim}{i} - (-2xyz^2 + 2y^2z)\underset{\sim}{j}$$

$$+ (2x - 3xy^2 + 2yz^2)\underset{\sim}{k}.$$

$$\text{At } (1,2,3), \quad curl\ \underset{\sim}{A} = -15\underset{\sim}{i} + 12\underset{\sim}{j} + 26\underset{\sim}{k}.$$

Remark

$\underset{\sim}{A}$ is a vector function and

$curl\ \underset{\sim}{A}$ is also a vector function.