

Surface Integral

Scalar Field, V Integral

If scalar field V exists on surface S , surface integral V of S is defined by

$$\int_S V d\tilde{S} = \int_S V \tilde{n} dS$$

where

$$\tilde{n} = \frac{\nabla S}{|\nabla S|}$$

Example :

Scalar field $V = x y z$ defined on the surface $S : x^2 + y^2 = 4$ between $z = 0$ and $z = 3$ in the first octant.

Evaluate $\int_S V dS$

Solution

Given $S : x^2 + y^2 = 4$, so $\text{grad } S$ is

$$\nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x \mathbf{i} + 2y \mathbf{j}$$

Also,

$$|\nabla S| = \sqrt{(2x)^2 + (2y)^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

Therefore,

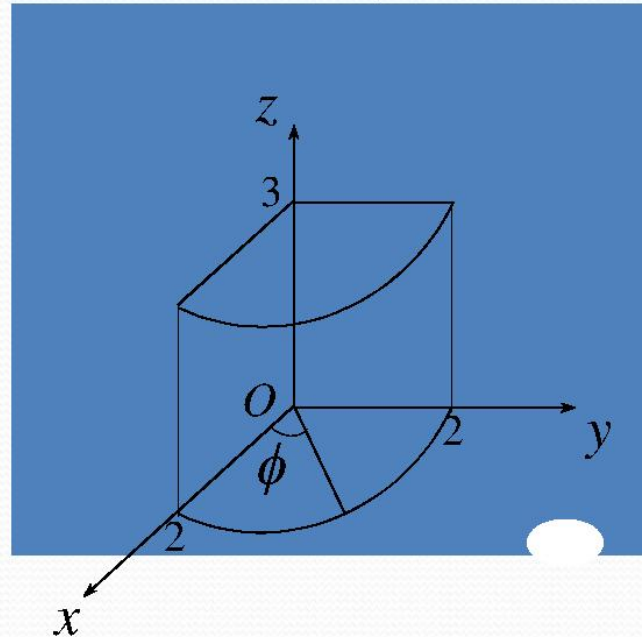
$$\vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\vec{i} + 2y\vec{j}}{4} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

Then,

$$\begin{aligned} \int_S V \vec{n} \, dS &= \int_S xyz \left(\frac{1}{2} \right) (x\vec{i} + y\vec{j}) \, dS \\ &= \frac{1}{2} \int (x^2 yz \vec{i} + xy^2 z \vec{j}) \, dS \end{aligned}$$

Surface $S : x^2 + y^2 = 4$ is bounded by $z = 0$ and $z = 3$ that is a cylinder with z -axis as a cylinder axes and radius, $\rho = \sqrt{4} = 2$.

So, we will use polar coordinate of cylinder to find the surface integral.



Polar Coordinate for Cylinder

$$x = \rho \cos \phi = 2 \cos \phi$$

$$y = \rho \sin \phi = 2 \sin \phi$$

$$z = z$$

$$dS = \rho d\phi dz$$

where $0 \leq \phi \leq \frac{\pi}{2}$ (1st octant) and $0 \leq z \leq 3$

Using polar coordinate of cylinder,

$$x^2 yz = (2 \cos \phi)^2 (2 \sin \phi) z = 8z \cos^2 \phi \sin \phi$$

$$xy^2 z = (2 \cos \phi)(2 \sin \phi)^2 (z) = 8z \sin^2 \phi \cos \phi$$

From

$$\int_S V \underline{\tilde{n}} dS = \frac{1}{2} \int_S (x^2 yz \underline{\tilde{i}} + xy^2 z \underline{\tilde{j}}) dS = \int_S V d \underline{\tilde{S}}$$

Therefore,

$$\begin{aligned}\int_S \vec{V} d\vec{S} &= \frac{1}{2} \int_{\phi=0}^{\pi/2} \int_{z=0}^3 (8z \cos^2 \phi \sin \phi \vec{i} + 8z \sin^2 \phi \cos \phi \vec{j})(2) dz d\phi \\ &= 8 \int_0^{\pi/2} \left[\frac{1}{2} z^2 \cos^2 \phi \sin \phi \vec{i} + \frac{1}{2} z^2 \sin^2 \phi \cos \phi \vec{j} \right]_0^3 d\phi \\ &= 8 \int_0^{\pi/2} \left[\frac{9}{2} \cos^2 \phi \sin \phi \vec{i} + \frac{9}{2} \sin^2 \phi \cos \phi \vec{j} \right] d\phi \\ &= 8 \times \frac{9}{2} \int_0^{\pi/2} \left[\cos^2 \phi \sin \phi \vec{i} + \sin^2 \phi \cos \phi \vec{j} \right] d\phi \\ &= 36 \left[\frac{\cos^3 \phi \sin \phi}{3(-\sin \phi)} \vec{i} + \frac{\sin^3 \phi \cos \phi}{3(\cos \phi)} \vec{j} \right]_0^{\pi/2} \\ &= 12(\vec{i} + \vec{j})\end{aligned}$$

Exercise:

If V is a scalar field where $V = xyz^2$, evaluate

*$\int_S V \, d\vec{S}$ for surface S that region bounded by $x^2 + y^2 = 9$
between $z = 0$ and $z = 2$ in the first octant.*

Answer: $24(\vec{i} + \vec{j})$

Vector Field, Integral

If vector field \vec{F} is defined on surface S , surface

integral \vec{F} of S is defined as

$$\int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n} dS$$

where $\vec{n} = \frac{\nabla S}{|\nabla S|}$

Example:

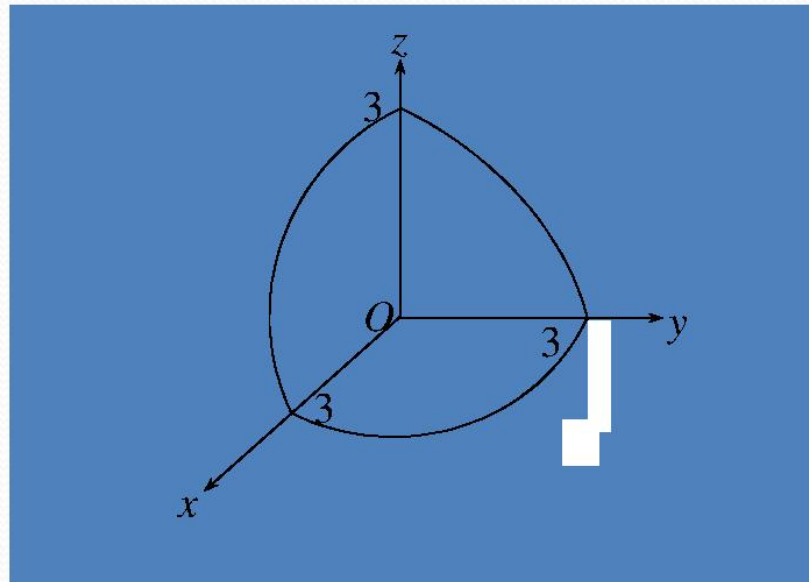
Vector field $\vec{F} = y\vec{i} + 2z\vec{j} + x\vec{k}$ defined on surface

$S : x^2 + y^2 + z^2 = 9$ and bounded by $x = 0, y = 0, z = 0$ in the first octant.

Evaluate $\int_S \vec{F} \cdot d\vec{S}$.

Solution

Given $S : x^2 + y^2 + z^2 = 9$ is bounded by $x = 0$, $y = 0$, $z = 0$ in the 1st octant. This refers to a sphere with center at $(0,0,0)$ and radius, $r = 3$, in the 1st octant.



So, grad S is

$$\begin{aligned}\nabla S &= \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} \\ &= 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k},\end{aligned}$$

and

$$\begin{aligned}|\nabla S| &= \sqrt{(2x)^2 + (2y)^2 + (2z)^2} \\ &= 2\sqrt{x^2 + y^2 + z^2} \\ &= 2\sqrt{9} = 6.\end{aligned}$$

$$\begin{aligned} \therefore \underline{\underline{n}} &= \frac{\nabla S}{|\nabla S|} = \frac{2x \underline{\underline{i}} + 2y \underline{\underline{j}} + 2z \underline{\underline{k}}}{6} \\ &= \frac{1}{3} (x \underline{\underline{i}} + y \underline{\underline{j}} + z \underline{\underline{k}}). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_S \underline{\underline{F}} \cdot d\underline{\underline{S}} &= \int_S \underline{\underline{F}} \cdot \underline{\underline{n}} dS \\ &= \int_S (y \underline{\underline{i}} + 2 \underline{\underline{j}} + \underline{\underline{k}}) \left(\frac{1}{3} \right) (x \underline{\underline{i}} + y \underline{\underline{j}} + z \underline{\underline{k}}) dS \\ &= \frac{1}{3} \int_S (xy + 2y + z) dS. \end{aligned}$$

Using polar coordinate of sphere,

$$x = r \sin \theta \cos \phi = 3 \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi = 3 \sin \theta \sin \phi$$

$$z = r \cos \theta = 3 \cos \theta$$

$$dS = r^2 \sin \theta d\theta d\phi = 9 \sin \theta d\theta d\phi$$

$$\text{where } 0 \leq \theta, \phi \leq \frac{\pi}{2}.$$

$$\begin{aligned}
\therefore \int_S \vec{F} \cdot d\vec{S} &= \frac{1}{3} \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} [(3 \sin \theta \cos \phi)(3 \sin \theta \sin \phi) \\
&\quad + 2(3 \sin \theta \sin \phi) + 3 \cos \theta][9 \sin \theta] d\theta d\phi \\
&= 9 \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} [3 \sin^3 \theta \sin \phi \cos \phi \\
&\quad + 2 \sin^2 \theta \sin \phi + \sin \theta \cos \theta] d\theta d\phi
\end{aligned}$$

⋮

$$= 9 \left(1 + \frac{3\pi}{4} \right)$$

Exercise:

Evaluate $\int_S \vec{F} \cdot d\vec{S}$ on S , where $\vec{F} = x\vec{i} + 2z\vec{j} + y\vec{k}$

and S is a surface of the region bounded by

$x^2 + y^2 + z^2 = 4$, $x = 0$, $y = 0$ and $z = 0$ in the 1st octant.

$$\text{Answer: } 8 \left(\frac{\pi}{6} + 1 \right)$$