Green's Theorem

If c is a closed curve in counter-clockwise on plane-xy, and given two functions $P(x, y)$ and $Q(x, y)$,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_c (P\,dx + Q\,dy)$$

where S is the area of c.

Example :

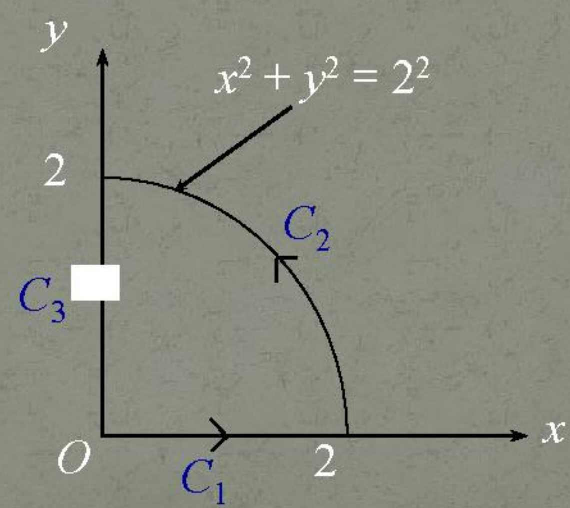
Prove Green's Theorem for

$$\ldots + (x+2y)dy]$$

which has been evaluated by boundary that defined as

$x=0, y=0$ *and* $x^2+y^2=4$ in the first quarter.

Solution



Given [illegible] $+(x+2y)dy]$ where

$P = x^2 + y^2$ and $Q = x + 2y$. We defined curve c

as c_1, c_2 *and* c_3.

i) For $c_1 : y = 0,\ dy = 0\ and\ \ 0 \le x \le 2$

$$\int_{c_1} (Pdx + Qdy) = \int_{c_1} \left[(x^2 + y^2)dx + (x + 2y)dy \right]$$

$$= \int_0^2 x^2 dx$$

$$= \left[\frac{1}{3} x^3 \right]_0^2 = \frac{8}{3}.$$

ii) For $c_2 : x^2 + y^2 = 4$, in the first quarter from (2,0) to (0,2).

This curve actually a part of a circle.

Therefore, it's more easier if we integrate by using polar coordinate of plane,

$$x = 2\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{2}$$

$$\Rightarrow dx = -2\sin\theta\, d\theta,\ dy = 2\cos\theta\, d\theta.$$

$$\int_{c_2} (Pdx + Qdy) = \int_{c_2} \left[(x^2 + y^2)dx + (x + 2y)dy\right]$$

$$= \int_0^{\frac{\pi}{2}} [((2\cos\theta)^2 + (2\sin\theta)^2)(-2\sin\theta\, d\theta)$$

$$+ ((2\cos\theta + 2(2\sin\theta))(2\cos\theta\, d\theta)]$$

$$= \int_0^{\frac{\pi}{2}} (-8\sin\theta + 4\cos^2\theta + 8\sin\theta\cos\theta)d\theta$$

$$= \int_0^{\frac{\pi}{2}} (-8\sin\theta + 2 + 2\cos 2\theta + 8\sin\theta\cos\theta)d\theta$$

$$= \left[8\cos\theta + 2\theta + \sin 2\theta + 4\sin^2\theta\right]_0^{\frac{\pi}{2}}$$

$$= -8 + \pi + 4 \quad = \pi - 4.$$

iii) $\quad For\ c_3 : x = 0,\ dx = 0,\ 0 \le y \le 2$

$$\int_{c_3} (Pdx + Qdy) = \int_{c_3} \left[(x^2 + y^2)dx + (x + 2y)dy \right]$$

$$= \int_2^0 2y\,dy$$

$$= \left[y^2 \right]_2^0$$

$$= -4.$$

$$\therefore \int_c (Pdx + Qdy) = \frac{8}{3} + (\pi - 4) - 4 = \pi - \frac{16}{3}.$$

b) Now, we evaluate $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$

where $\frac{\partial Q}{\partial x} = 1 \; and \; \frac{\partial P}{\partial y} = 2y.$

Again,because this is a part of the circle,

we shall integrate by using polar coordinate of plane,

$$x = r\cos\theta, \; y = r\sin\theta$$

where $0 \le \mathrm{r} \le 2, \; 0 \le \theta \le \frac{\pi}{2} \quad and \quad dxdy = dS = r\,dr\,d\theta.$

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_S (1-2y)\,dx\,dy$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} (1-2r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2} r^2 - \frac{2}{3} r^3 \sin\theta \right]_0^2 d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left(2 - \frac{16}{3} \sin\theta \right) d\theta$$

$$= \left[2\theta + \frac{16}{3} \cos\theta \right]_0^{\frac{\pi}{2}}$$

$$= \pi - \frac{16}{3}.$$

Therefore,

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \pi - \frac{16}{3}.$$

$$LHS = RHS$$

$\Rightarrow$ Green's Theorem has been proved.