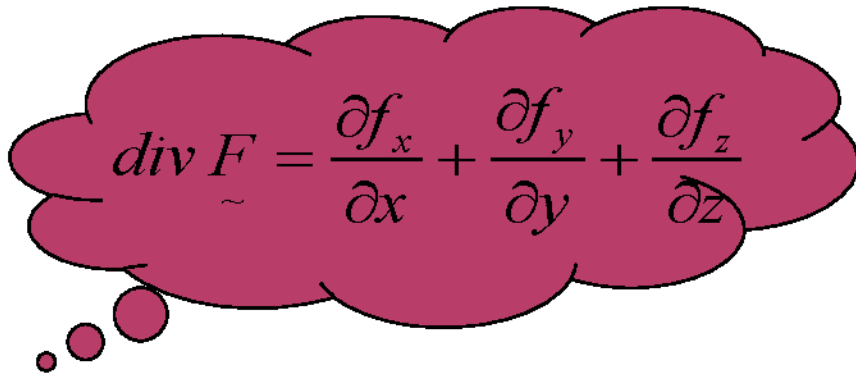


DIVERGENCE THEOREM (GAUSS' THEOREM)

If S is a closed surface including region V in vector field \vec{F}

$$\int_V \operatorname{div} \vec{F} dV = \int_S \vec{F} \cdot d\vec{S}.$$


$$\operatorname{div} \vec{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

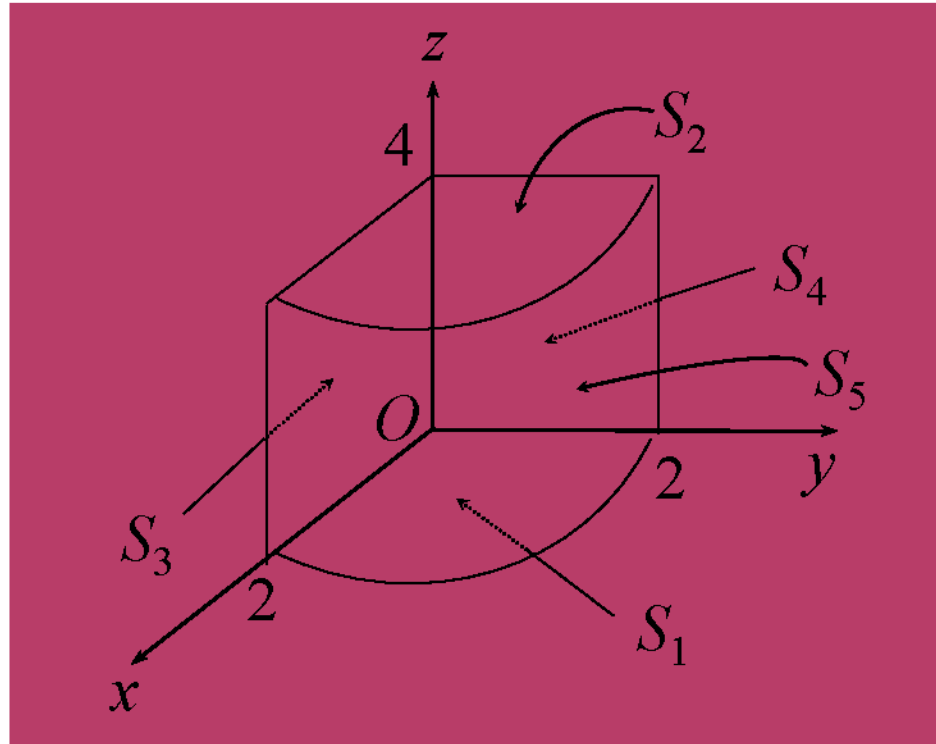
EXAMPLE:

Prove Gauss' Theorem for vector field,

$$\vec{F} = x \vec{i} + 2y \vec{j} + z^2 \vec{k} \quad \text{in the region bounded by}$$

planes $z = 0, z = 4, x = 0, y = 0$ and $x^2 + y^2 = 4$
in the first octant.

Solution



For this problem, the region of integration is bounded by 5 planes :

$$S_1 : z = 0$$

$$S_2 : z = 4$$

$$S_3 : y = 0$$

$$S_4 : x = 0$$

$$S_5 : x^2 + y^2 = 4$$

To prove Gauss' Theorem, we evaluate both $\int_V \operatorname{div} \underline{F} dV$

and $\int_S \underline{F} \cdot d\underline{S}$,

The answer should be the same.

1) We evaluate $\int_V \text{div } \vec{F} dV$. Given $\vec{F} = x\vec{i} + 2y\vec{j} + z^2\vec{k}$.

So,

$$\begin{aligned}\text{div } \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z^2) \\ &= 1 + 2z.\end{aligned}$$

$$\text{Also, } \int_V \text{div } \vec{F} dV = \int_V (1 + 2z) dV.$$

The region is a part of the cylinder. So, we integrate by using polar coordinate of cylinder ,

$$\begin{aligned}x &= \rho \cos \phi; & y &= \rho \sin \phi; & z &= z \\ dV &= \rho d\rho d\phi dz\end{aligned}$$

where $0 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq z \leq 4$.

Therefore,

$$\begin{aligned}\int_V (1 + 2z) dV &= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=0}^2 \int_{z=0}^4 (1 + 2z) \rho dz d\rho d\phi \\ &= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=0}^2 \rho [z + z^2]_0^4 d\rho d\phi \\ &= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=0}^2 (20\rho) d\rho d\phi \\ &= \int_{\phi=0}^{\frac{\pi}{2}} [10\rho^2]_0^2 d\phi \\ &= \int_{\phi=0}^{\frac{\pi}{2}} (40) d\phi \\ &= [40\phi]_0^{\frac{\pi}{2}} \\ &= 20\pi.\end{aligned}$$

$$\therefore \int_V \operatorname{div} \tilde{F} dV = 20\pi.$$

2) Now, we evaluate $\int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n} dS$.

i) S_1 : $z = 0, \vec{n} = -\vec{k}, dS = r dr d\theta$

$$\Rightarrow \vec{F} = x\vec{i} + 2y\vec{j} + 0\vec{k}$$

$$\Rightarrow \vec{F} \cdot \vec{n} = (x\vec{i} + 2y\vec{j}) \cdot (-\vec{k}) = 0$$

$$\therefore \int_{S_1} \vec{F} \cdot \vec{n} dS = 0.$$

$$\text{ii) } S_2 : \quad z = 4, \quad \underline{n} = \underline{k}, \quad dS = r dr d\theta$$

$$\Rightarrow \underline{F} = x \underline{i} + 2 \underline{j} + (4)^2 \underline{k} = x \underline{i} + 2 \underline{j} + 16 \underline{k}$$

$$\Rightarrow \underline{F} \cdot \underline{n} = (x \underline{i} + 2 \underline{j} + 16 \underline{k}) \cdot (\underline{k}) = 16.$$

Therefore for S_2 , $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned} \therefore \int_{S_2} \underline{F} \cdot \underline{n} dS &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^2 16 r dr d\theta \\ &= \dots \\ &= 16\pi. \end{aligned}$$

$$\text{iii) } S_3 : \quad y = 0, \quad \underline{n} = -\underline{j}, \quad dS = dx dz$$

$$\Rightarrow \underline{F} = x \underline{i} + 2 \underline{j} + z^2 \underline{k}$$

$$\begin{aligned} \Rightarrow \underline{F} \cdot \underline{n} &= (x \underline{i} + 2 \underline{j} + z^2 \underline{k}) \cdot (-\underline{j}) \\ &= -2. \end{aligned}$$

Therefore for S_3 , $0 \leq x \leq 2$, $0 \leq z \leq 4$

$$\begin{aligned} \therefore \int_{S_3} \underline{F} \cdot \underline{n} dS &= \int_{x=0}^2 \int_{z=0}^4 (-2) dz dx \\ &= \dots \\ &= -16. \end{aligned}$$

$$\text{iv) } S_4 : \quad x = 0, \quad \underline{\underline{n}} = -\underline{\underline{i}}, \quad dS = dydz$$

$$\Rightarrow \underline{\underline{F}} = 0\underline{\underline{i}} + 2\underline{\underline{j}} + z^2\underline{\underline{k}} = 2\underline{\underline{j}} + z^2\underline{\underline{k}}$$

$$\Rightarrow \underline{\underline{F}} \cdot \underline{\underline{n}} = (2\underline{\underline{j}} + z^2\underline{\underline{k}}) \cdot (-\underline{\underline{i}}) = 0.$$

$$\therefore \int_{S_4} \underline{\underline{F}} \cdot \underline{\underline{n}} dS = 0.$$

$$v) S_5: \quad x^2 + y^2 = 4, \quad dS = \rho d\phi dz$$

$$\nabla S_5 = 2x \underline{\tilde{i}} + 2y \underline{\tilde{j}} \quad \text{and} \quad |\nabla S_5| = 4$$

$$\begin{aligned} \Rightarrow \underline{\tilde{n}} &= \frac{\nabla S_5}{|\nabla S_5|} = \frac{2x \underline{\tilde{i}} + 2y \underline{\tilde{j}}}{4} \\ &= \frac{1}{2}(x \underline{\tilde{i}} + y \underline{\tilde{j}}). \end{aligned}$$

By using polar coordinate of cylinder :

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$$

where for S_5 :

$$\rho = 2, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq z \leq 4, \quad dS = 2d\phi dz$$

$$\begin{aligned}
\Rightarrow \vec{F} \cdot \vec{n} &= (x \vec{i} + 2y \vec{j} + z^2 \vec{k}) \cdot \left(\frac{1}{2} x \vec{i} + \frac{1}{2} y \vec{j} \right) \\
&= \frac{1}{2} x^2 + y \\
&= \frac{1}{2} (\rho \cos \phi)^2 + (\rho \sin \phi) \\
&= 2 \cos^2 \phi + 2 \sin \phi; \quad \text{kerana } \rho = 2. \\
&= 2(\cos^2 \phi + \sin \phi).
\end{aligned}$$

$$\begin{aligned}
\therefore \int_{S_5} \vec{F} \cdot \vec{n} dS &= \int_{\phi=0}^{\frac{\pi}{2}} \int_{z=0}^4 (2)(\cos^2 \phi + \sin \phi)(2) d\phi dz \\
&= \dots \\
&= 16 + 4\pi.
\end{aligned}$$

Finally,

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{S} &= \int_{S_1} \vec{F} \cdot d\vec{S} + \int_{S_2} \vec{F} \cdot d\vec{S} + \int_{S_3} \vec{F} \cdot d\vec{S} + \int_{S_4} \vec{F} \cdot d\vec{S} + \int_{S_5} \vec{F} \cdot d\vec{S} \\ &= 0 + 16\pi - 16 + 0 + 16 + 4\pi \\ &= 20\pi.\end{aligned}$$

$$\therefore \int_S \vec{F} \cdot d\vec{S} = 20\pi.$$

LHS = RHS

\Rightarrow Gauss' Theorem has been proved.