ECE 6382

Legendre Functions

Wave Equation in Spherical Coordinates

Source - free scalar wave equation in spherical coordinates (r,θ,ϕ):

$$\nabla^2\psi + k^2\psi = 0, \qquad k = 2\pi/\lambda$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0$$

Assume $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$. Then separation of variables yields

- $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[(kr)^2 - n(n+1)\right]R = 0$ (spherical Bessel's equation)
- $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$ (Legendre's equation)
- $\Phi''(\phi) + m^2\Phi(\phi) = 0$ (Harmonic equation)

Solution of the Spherical Wave Equation

- $$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[(kr)^2-n(n+1)\right]R=0 \quad\Rightarrow\quad R(r)=z_n(kr)$$

  where $z_n(kr)=j_n(kr),\ n_n(kr), h_n^{(1)}(kr), h_n^{(2)}(kr),$

- $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \quad\Rightarrow\quad \Theta(\theta)=L_n^m(\cos\theta),$$

  where $L_n^m(\cos\theta)=P_n^m(\cos\theta),\ Q_n^m(\cos\theta)$, n an integer if $0\le\theta\le\pi$

- $\Phi''(\phi)+m^2\Phi(\phi)=0 \quad\Rightarrow\quad \Phi(\phi)=e^{im\phi}$, m an integer if $0\le\phi\le 2\pi$

  $$\Rightarrow\quad \boxed{\psi(r,\theta,\phi)=R(\rho)\Theta(\theta)\Phi(\phi)=z_n(k\rho)\,L_n^m(\cos\theta)e^{im\phi}}$$

A superposition of these solutions over the allowable separation constants is

$$\boxed{\psi(r,\theta,\phi)=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}a_{mn}\,z_n(k\rho)\,L_n^m(\cos\theta)e^{im\phi}}$$

Legendre's Equation

The separated equation in the angular variable $x = \cos\theta$:

- $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + \left[n\left(n+1\right) - \dfrac{m^2}{1-x^2}\right]y \; = \; 0$ or
- $\dfrac{d}{dx}\left(\left(1-x^2\right)\dfrac{dy}{dx}\right) + \left[n\left(n+1\right) - \dfrac{m^2}{1-x^2}\right]y = 0$ (self - adjoint form)

For $m = 0$, this is simply *Legendre's equation*;

for $m \neq 0$, it is the *associated Legendre's equation*

Series Solution of Legendre's Equation

- $$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!(n-2k)!(n-k)!}$$

where $\lfloor \frac{n}{2} \rfloor$, the "floor" of $\frac{n}{2}$, is the largest integer contained in $\frac{n}{2}$:

$$\lfloor \tfrac{n}{2} \rfloor = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd} \end{cases}$$

- A second solution is given by

$$Q_n(x) = P_n(x)\left[\frac{1}{2}\ln\frac{1+x}{1-x} - \Phi(n)\right] + \sum_{m=1}^{n} \frac{(-1)^m (n+m)!\, x^{n-2k}}{(m!)^2 (n-m)!} \Phi(m) \left(\frac{1-x}{2}\right)^m$$

where $\Phi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$ *Only* $P_n(x)$ *is finite at* $x = \pm 1$.

Low Order Legendre Functions

$$L_n(x) = P_n(x),\ Q_n(x)$$

	$P_n(x)$	$P_n(\cos\theta)$	$Q_n(x)$	$Q_n(\cos\theta)$
$n=0:$	1	1	$\frac{1}{2}\ln\frac{1+x}{1-x}$	$\frac{1}{2}\ln\cot\frac{\theta}{2}$
$n=1:$	x	$\cos\theta$	$\frac{x}{2}\ln\frac{1+x}{1-x}-1$	$\cos\theta\ln\cot\frac{\theta}{2}-1$
$n=2:$	$\frac{1}{2}\left(3x^2-1\right)$	$\frac{1}{4}\left(3\cos2\theta+1\right)$	$\frac{3x^2-1}{4}\ln\frac{1+x}{1-x}-\frac{3x}{2}$	$\frac{3\cos^2\theta-1}{2}\ln\cot\frac{\theta}{2}-\frac{3\cos\theta}{2}$

Generating Function

The (integer order) Legendre functions can also be defined through a *generating function* $g(x,t)$:

- $$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n, \quad |t| \le 1$$

The generating function definition also leads to the integral representation

- $$P_n(x) = \frac{1}{2\pi i}\oint_C \frac{t^{-n-1}}{\sqrt{1-2xt+t^2}}\,dt$$

Recurrence Relations

Several recurrence relations are easily derived from the generating function. For example, from $g(x,t) = \dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$,

we note that

$$\frac{\partial}{\partial t} g(x,t) = \frac{\partial}{\partial t} \frac{1}{\sqrt{1-2xt+t^2}}$$

$$= \frac{x-t}{\left(1-2xt+t^2\right)^{\frac{3}{2}}} = \frac{x-t}{\left(1-2xt+t^2\right)} g(x,t) = \sum_{n=-\infty}^{\infty} n\, P_n(x)\, t^{n-1}$$

Rearranging and equating like powers of t yields

$$\boxed{(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)}$$

Recurrence Relations (cont.)

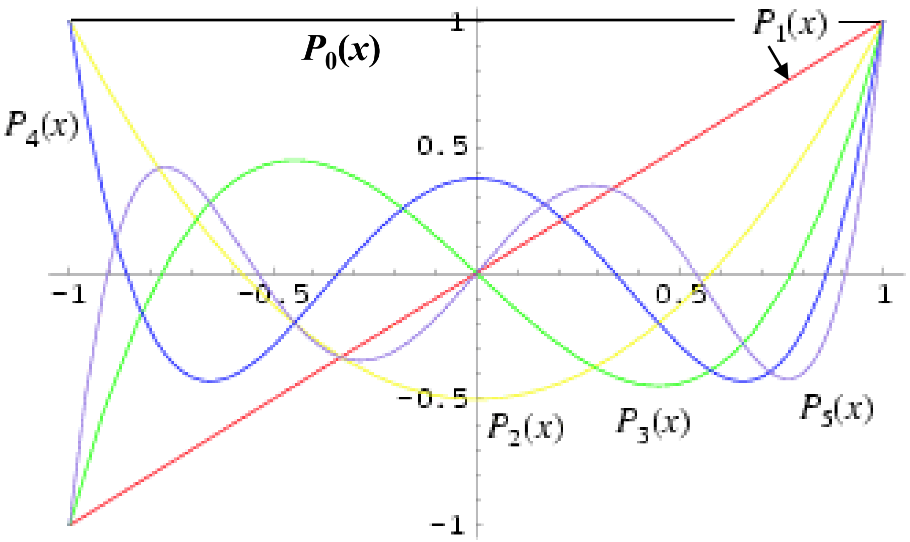
For $L_n(x) = P_n(x)$ or $Q_n(x)$:

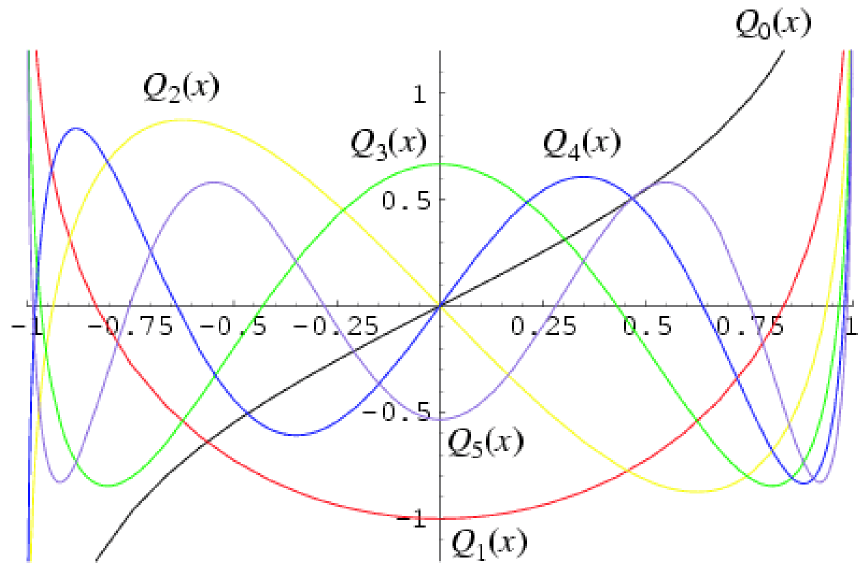
- $nL_n(x) = (2n-1)xL_{n-1}(x) - (n-1)L_{n-2}(x)$
- $xL_n'(x) - L_{n-1}'(x) = nL_n(x)$
- $L_n'(x) - xL_{n-1}'(x) = nL_{n-1}(x)$
- $(1-x)^2 L_n'(x) = nL_{n-1}(x) - nxL_n(x)$
- $(2n+1)L_n(x) = L_{n+1}'(x) - L_{n-1}'(x)$
- $(1-x)^2 L_{n-1}'(x) = nxL_{n-1}(x) - nL_n(x)$

Wronskians

- $$W[P_\nu^\mu, Q_\nu^\mu; x] = P_\nu^\mu(x)\frac{dQ_\nu^\mu(x)}{dx} - Q_\nu^\mu(x)\frac{dP_\nu^\mu(x)}{dx}$$
$$= \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}(1-x^2)^{-1}$$

Plots of Legendre Functions





Special Values and Relations

- $P_n(1) = 1,$
- $Q_n(1) = \infty$
- $P_n(-1) = (-1)^n,$
- $Q_n(-1) = \infty$
- $P_n(-x) = (-1)^n P_n(x)$
- $P_n(x)$ is an nth degree polynomial with n zeros in $-1 \le x \le 1$

- Rodrigue's formula :

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Solutions of the Associated Legendre Equation

- $P_n^0(x) \equiv P_n(x),$
- $Q_n^0(x) \equiv Q_n(x),$
- $P_n^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^m P_n(x)}{dx^m},$
- $Q_n^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^m Q_n(x)}{dx^m}$
- $P_n^m(x) = Q_n^m(x) = 0, \quad m > n, \quad m, n$ integers

Recurrence Relations for Associated Legendre Functions

For $L_n^m = P_n^m$ or Q_n^m,

- $(m-n-1)L_{n+1}^m + (2n+1)xL_n^m - (m+n)L_{n-1}^m = 0$ (recursion on n)
- $L_n^{m+1} + \dfrac{2mx}{\left(1-x^2\right)^{\frac{1}{2}}} L_n^m + (m+n)(n-m+1)L_n^{m-1} = 0$ (recursion on m)
- $L_n^m{}' = \dfrac{1}{\left(1-x^2\right)}\left[-nxL_n^m + (m+n)L_{n-1}^m\right]$
- $L_n^m{}' = \dfrac{1}{\left(1-x^2\right)}\left[(n+1)xL_n^m - (n-m+1)L_{n+1}^m\right]$
- $L_n^m{}' = \dfrac{mx}{\left(1-x^2\right)} L_n^m + \dfrac{(n+m)(n-m+1)}{\left(1-x^2\right)^{\frac{1}{2}}} L_n^{m-1}$
- $L_n^m{}' = -\dfrac{mx}{\left(1-x^2\right)} L_n^m - \dfrac{1}{\left(1-x^2\right)^{\frac{1}{2}}} L_n^{m+1}$

Special Values of the Associated Legendre Functions

- $P_n^m(1) = \begin{cases} 1, & m = 0 \\ 0, & m > 0 \end{cases}, \qquad Q_n^m(1) = \infty,$
- $P_n^m(0) = \begin{cases} (-1)^{(n+m)/2} \, 2^{(n-m)/2} \left(\dfrac{n-m}{2}\right)!, & n+m \text{ even} \\ 0, & n+m \text{ odd} \end{cases},$
- $\left[\dfrac{d^r}{dx^r} P_n^m(x)\right]_{x=0} = (-1)^r \, P_n^{m+r}(0)$
- $\left[\dfrac{d^r}{dx^r} Q_n^m(x)\right]_{x=0} = (-1)^r \, Q_n^{m+r}(0)$

Orthogonalities

- $$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx$$

$$= \int_0^{\pi} P_p^m(\cos\theta) P_q^m(\cos\theta) \sin\theta d\theta \;=\; \frac{2}{2q+1}\frac{(q+m)!}{(q-m)!}\delta_{pq}$$

- $$\int_{-1}^{1} \frac{P_p^m(x) P_p^n(x)}{1-x^2}\, dx$$

$$= \int_0^{\pi} \frac{P_p^m(\cos\theta) P_p^n(\cos\theta)}{\sin\theta} d\theta \;=\; \frac{1}{m}\frac{(p+m)!}{(p-m)!}\delta_{mn}$$

- $$\int_0^{\pi} \left(\frac{dP_p^m(\cos\theta)}{d\theta}\frac{dP_q^m(\cos\theta)}{d\theta} + \frac{m^2}{\sin^2\theta} P_p^m(\cos\theta) P_q^m(\cos\theta) \right) \sin\theta d\theta$$

$$= \frac{2}{2p+1}\frac{(p+m)!}{(p-m)!} p(p+1)\delta_{pq}$$

The Tesseral Harmonics and Their Orthogonalities

The *tesseral* (*spherical*) *harmonics* :

- $$Y_n^m(\theta,\phi) \equiv \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta)e^{im\phi}$$

Orthogonalities of the tesseral harmonics :

- $$\int_0^{2\pi}\int_0^{\pi} Y_p^m(\theta,\phi)Y_q^n(\theta,\phi)\sin\theta d\theta d\phi = \delta_{pq}\delta_{mn}$$

Tesseral Harmonic Expansion

Expansion of a function on a spherical surface :

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} Y_n^m(\theta,\phi)$$

$$\Rightarrow \quad a_{mn} = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi) Y_n^m(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

where

$$Y_n^m(\theta,\phi) \equiv \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta)\, e^{im\phi}$$