Ordinary Differential Equations

CHAPTER 1
Introduction to **Differential Equations**

1.1 Definitions and Terminology

1.2 Initial-Value Problems

1.3 Differential Equation as Mathematical Models

1.1 Definitions and Terminology

DEFINITION: differential equation

An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is said to be a **differential equation** (DE).

(Zill, Definition 1.1, page 6).

1.1 Definitions and Terminology

Recall ***Calculus***

Definition of a Derivative

If $y = f(x)$, the derivative of y or $f(x)$

With respect to x is defined as

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative is also denoted by y', $\frac{df}{dx}$ or $f'(x)$

1.1 Definitions and Terminology

Recall the Exponential function

$$y = f(x) = e^{2x}$$

➔dependent variable: y

➔independent variable: x

$$\frac{dy}{dx} = \frac{d(e^{2x})}{dx} = e^{2x}\left[\frac{d(2x)}{dx}\right] = 2e^{2x} = 2y$$

1.1 Definitions and Terminology

Differential Equation :

Equations that involve dependent variables and their derivatives with respect to the independent variables .

Differential Equations are classified by ***type***, ***order*** and ***linearity***.

1.1 Definitions and Terminology

Differential Equations are classified by ***type***, ***order*** and ***linearity***.

TYPE

There are two main *types* of differential equation: “ordinary” and “partial”.

1.1 Definitions and Terminology

Ordinary differential equation (ODE)
Differential equations that involve only **ONE** independent variable are called ordinary differential equations.

Examples:

$$\frac{dy}{dx} + 5y = e^x \text{ , } \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0 \text{, } \textbf{and } \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

➔only *ordinary* (or *total*) derivatives

1.1 Definitions and Terminology

Partial differential equation (PDE)
Differential equations that involve two or more independent variables are called partial differential equations.

Examples:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} \qquad \textbf{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

➔only *partial* derivatives

1.1 Definitions and Terminology

ORDER

The *order* of a differential equation is the order of the highest derivative found in the DE.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

second order **first order**

1.1 Definitions and Terminology

$xy' - y^2 = e^x$ ➔first order $F(x, y, y') = 0$

Written in differential form: $M(x, y)dx + N(x, y)dy = 0$

$y'' = x^3$ ➔second order $F(x, y, y', y'') = 0$

1.1 Definitions and Terminology

LINEAR or NONLINEAR

An *n*-th order differential equation is said to be **linear** if the function $F(x, y, y', \ldots, y^{(n)}) = 0$ is linear in the variables $y, \quad y', \ldots \quad y^{(n-1)}$

➔ $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

➔there are no multiplications among **dependent variables** and **their derivatives**. All **coefficients** are functions of **independent variables**.

A **nonlinear** ODE is one that is not linear, i.e. does not have the above form.

1.1 Definitions and Terminology

LINEAR or NONLINEAR

$$(y-x)dx+4xdy=0$$ or $$4x\frac{dy}{dx}+(y-x)=0$$

➔linear first-order ordinary differential equation

$$y''-2y'+y=0$$

➔linear second-order ordinary differential equation

$$\frac{d^3y}{dx^3}+3x\frac{dy}{dx}-5y=e^x$$

➔ linear third-order ordinary differential equation

1.1 Definitions and Terminology

LINEAR or NONLINEAR

$$(1-y)y' + 2y = e^x$$

coefficient depends on y

➔nonlinear first-order ordinary differential equation

$$\frac{d^2y}{dx^2} + \sin(y) = 0$$

nonlinear function of y

➔nonlinear second-order ordinary differential equation

$$\frac{d^4y}{dx^4} + y^2 = 0$$

power not 1

➔ nonlinear fourth-order ordinary differential equation

1.1 Definitions and Terminology

LINEAR or NONLINEAR

NOTE:

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots \qquad -\infty < x < \infty$$

$$\cos(y) = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots \qquad -\infty < x < \infty$$

1.1 Definitions and Terminology

Solutions of ODEs

DEFINITION: **solution of an ODE**

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous

on I, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

(Zill, Definition 1.1, page 8).

1.1 Definitions and Terminology

Namely, a solution of an *n*-th order ODE is a function which possesses at least *n* derivatives and for which

$$F(x,\ \phi(x),\ \phi^{'}(x),\ \phi^{(n)}(x)) = 0 \quad \text{for all } x \text{ in } I$$

We say that *satisfies* the differential equation on *I*.

1.1 Definitions and Terminology

Verification of a solution by substitution

Example: $y'' - 2y' + y = 0 \quad ; y = xe^{x}$

➔ $y' = xe^{x} + e^{x}, \quad y'' = xe^{x} + 2e^{x}$

➔left hand side:

$$y'' - 2y' + y = (xe^{x} + 2e^{x}) - 2(xe^{x} + e^{x}) + xe^{x} = 0$$

right-hand side: 0

The DE possesses the constant y=0 ➔trivial solution

1.1 Definitions and Terminology

DEFINITION: **solution curve**

A graph of the solution of an ODE is called a **solution curve, or an integral curve** of the equation.

1.1 Definitions and Terminology

DEFINITION: **families of solutions**

A solution containing an arbitrary constant (parameter) represents a set $G(x, y, c) = 0$ of solutions to an ODE called a **one-parameter family of solutions**.

A solution to an $n-th$ order ODE is a **n-parameter family of solutions** $F(x, y, y', \ldots\ldots y^{(n)}) = 0$.

Since the parameter can be assigned an infinite number of values, an ODE can have an infinite number of solutions.

1.1 Definitions and Terminology

Verification of a solution by substitution

Example: $y' + y = 2$

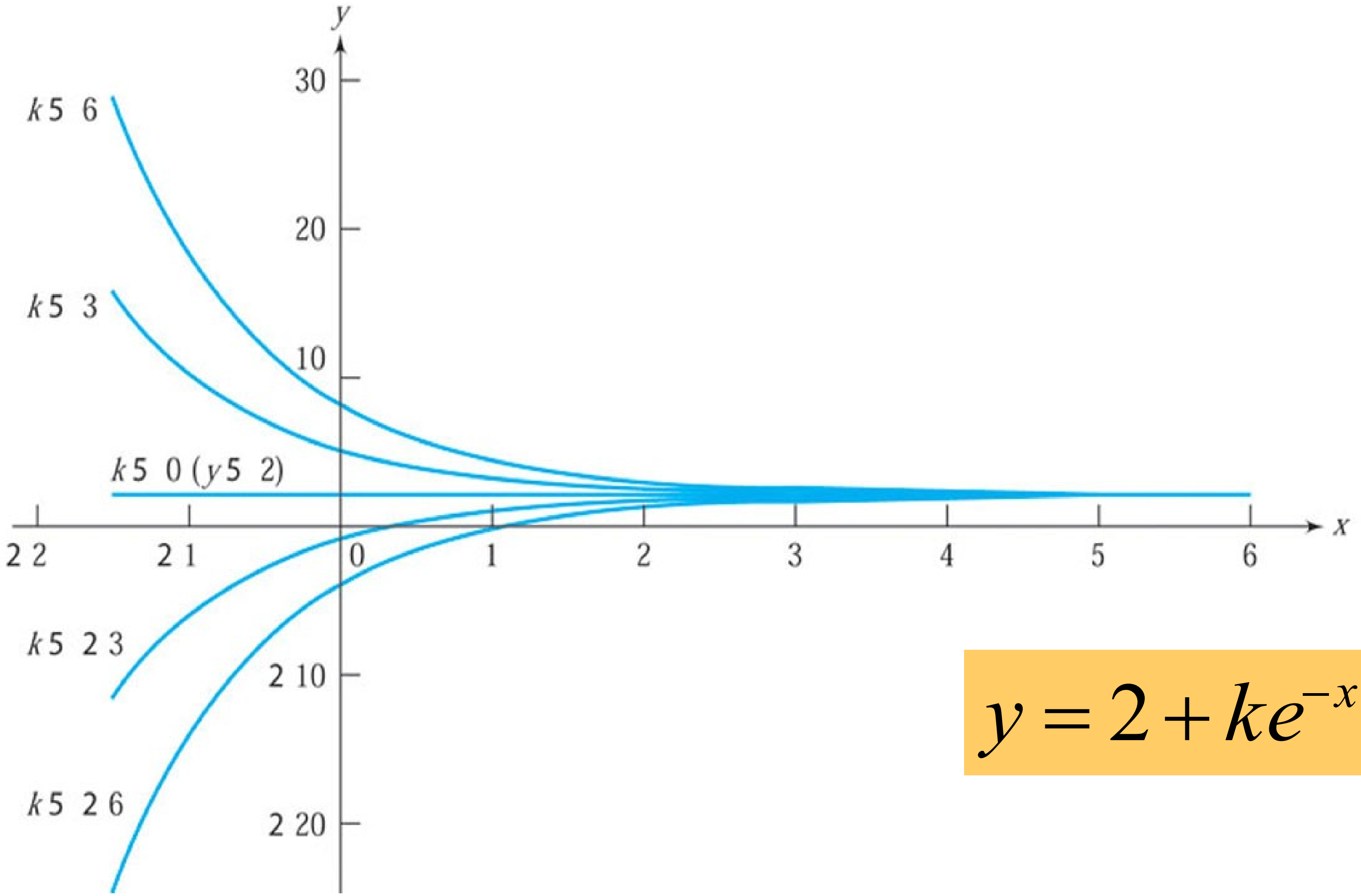
➔ $\varphi(x) = 2 + ke^{-x}$

$$y' + y = 2$$

$$\varphi(x) = 2 + ke^{-x}$$

$$\varphi'(x) = -ke^{-x}$$

$$\varphi'(x) + \varphi(x) = -ke^{-x} + 2 + ke^{-x} = 2$$



$$y = 2 + ke^{-x}$$

Figure 1.1 Integral curves of $y' + y = 2$ *for* k = 0, 3, –3, 6, *and* –6.

1.1 Definitions and Terminology

Verification of a solution by substitution

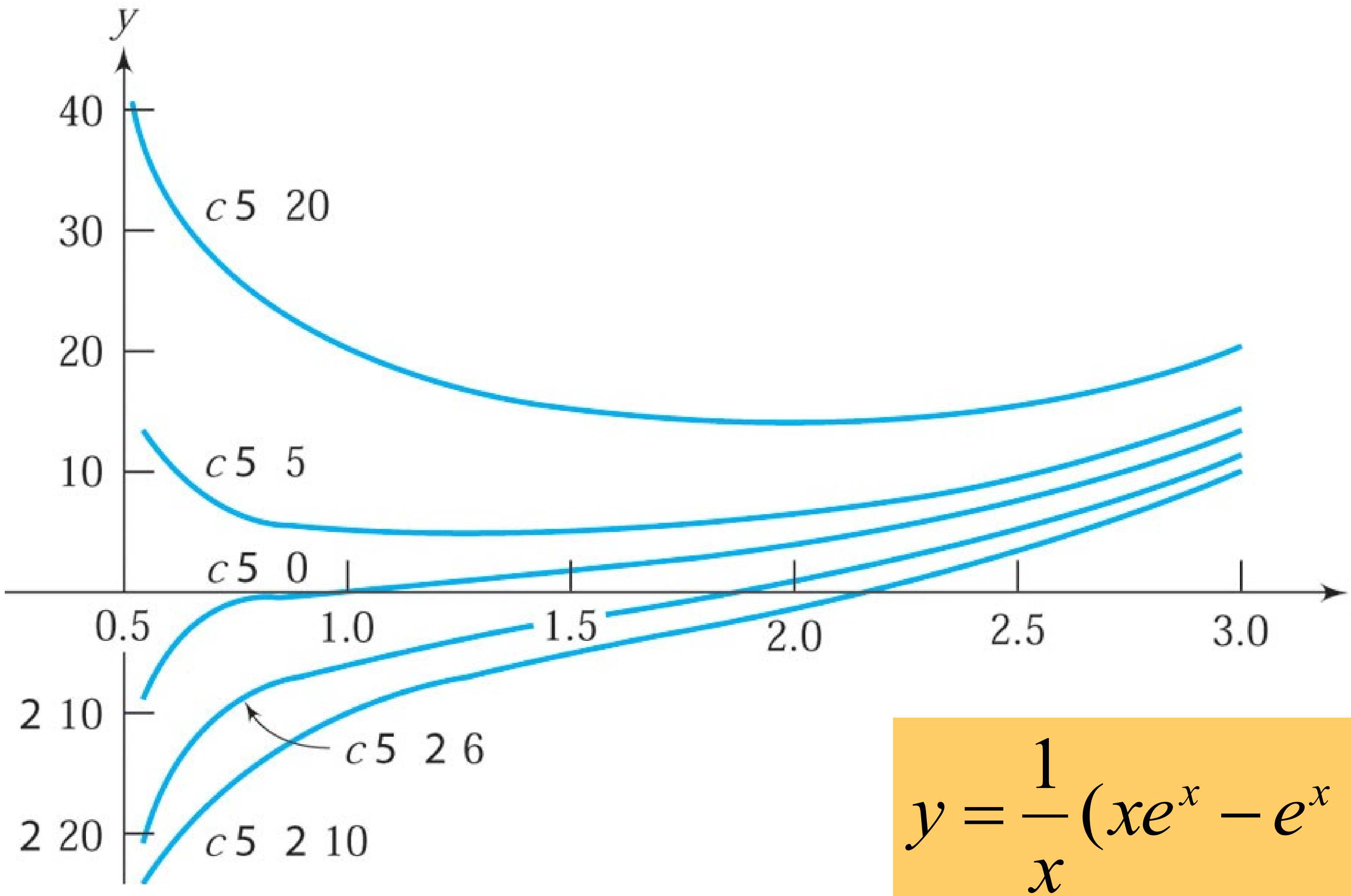
Example:

$$y' = \frac{y}{x} + 1$$

$$\varphi(x) = x\ln(x) + Cx$$ **for all** $x > 0$

$$\varphi'(x) = \ln(x) + 1 + C$$

$$\varphi'(x) = \frac{x\ln(x) + Cx}{x} + 1 = \frac{\varphi(x)}{x} + 1$$



$$y = \frac{1}{x}(xe^x - e^x + c)$$

Figure 1.2 ***Integral curves of*** $y' + \frac{1}{x}y = e^x$ ***for*** c **=0,5,20, -6,** ***and*** **–10.**

Second-Order Differential Equation

Example: $\varphi(x) = 6\cos(4x) - 17\sin(4x)$

is a solution of $y'' + 16x = 0$

By substitution:

$$\varphi' = -24\sin(4x) - 68\cos(4x)$$

$$\varphi'' = -96\cos(4x) + 272\sin(4x)$$

$$\varphi'' + 16\varphi = 0$$

$$F(x, y, y', y'') = 0$$

$$F\left(x, \varphi(x), \varphi'(x), \varphi(x)''\right) = 0$$

Second-Order Differential Equation

Consider the simple, linear second-order equation

$$y'' - 12x = 0$$

➔ $y'' = 12x \quad , \quad y' = \int y''(x)dx = \int 12x dx = 6x^2 + C$

➔ $y = \int y'(x)dx = \int (6x^2 + C)dx = 2x^3 + Cx + K$

To determine C and K, we need **two** initial conditions, one specify **a point** lying on the **solution curve** and the other its **slop**e at that point, e.g. $y(0) = K \quad , \quad y'(0) = C$

WHY ???

Second-Order Differential Equation

$$y'' = 12x$$

$$y = 2x^3 + Cx + K$$

IF only try $x=x_1$, and $x=x_2$

$$\rightarrow \quad y(x_1) = 2x_1^{\,3} + Cx_1 + K$$

$$y(x_2) = 2x_2^{\,3} + Cx_2 + K$$

It cannot determine C and K,

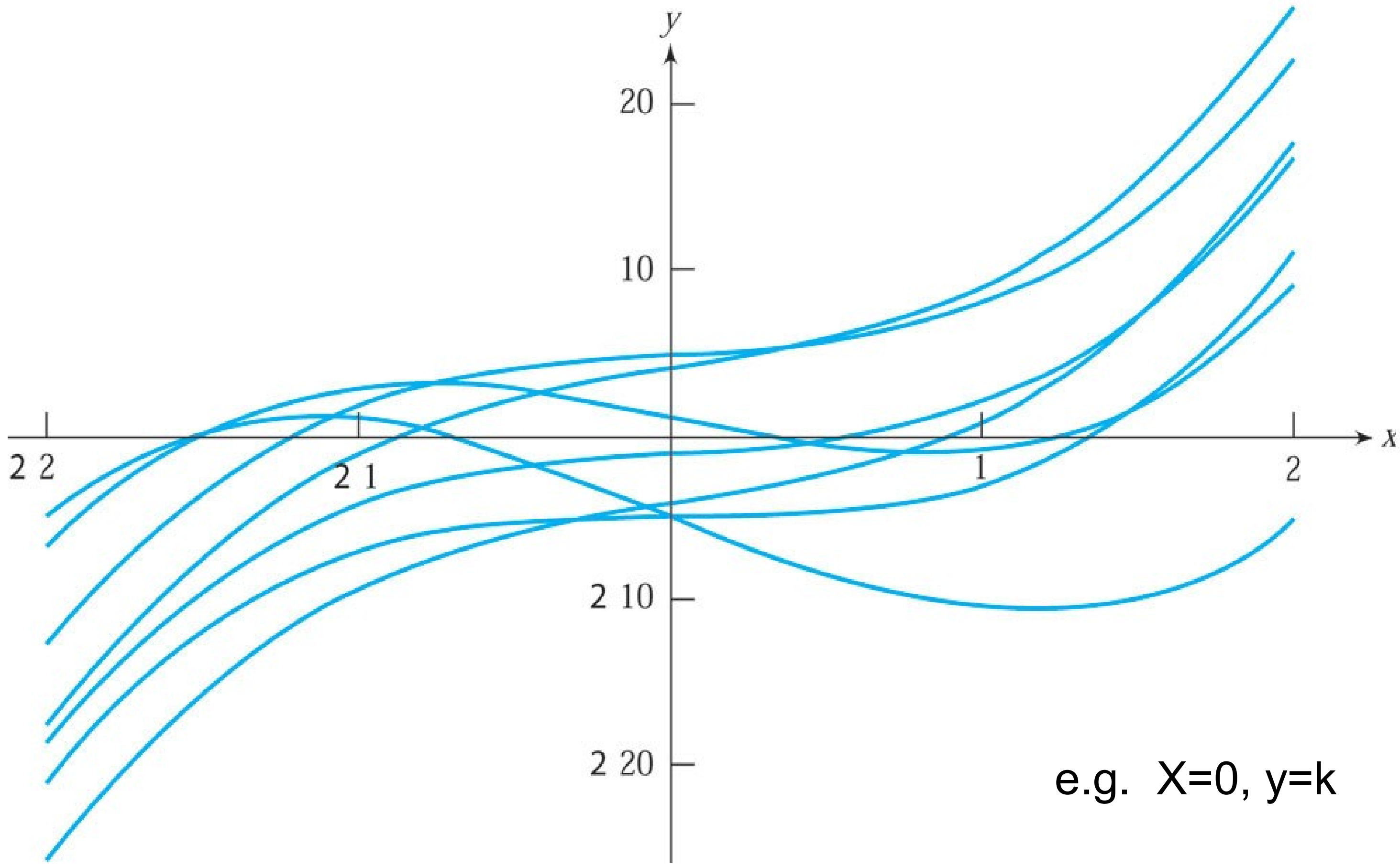


Figure 2.1 ***Graphs of*** $y = 2x^3 + Cx + K$ ***for various values of C and K.***

To satisfy the I.C. $y(0)=3$
The solution curve must pass through (0,3)

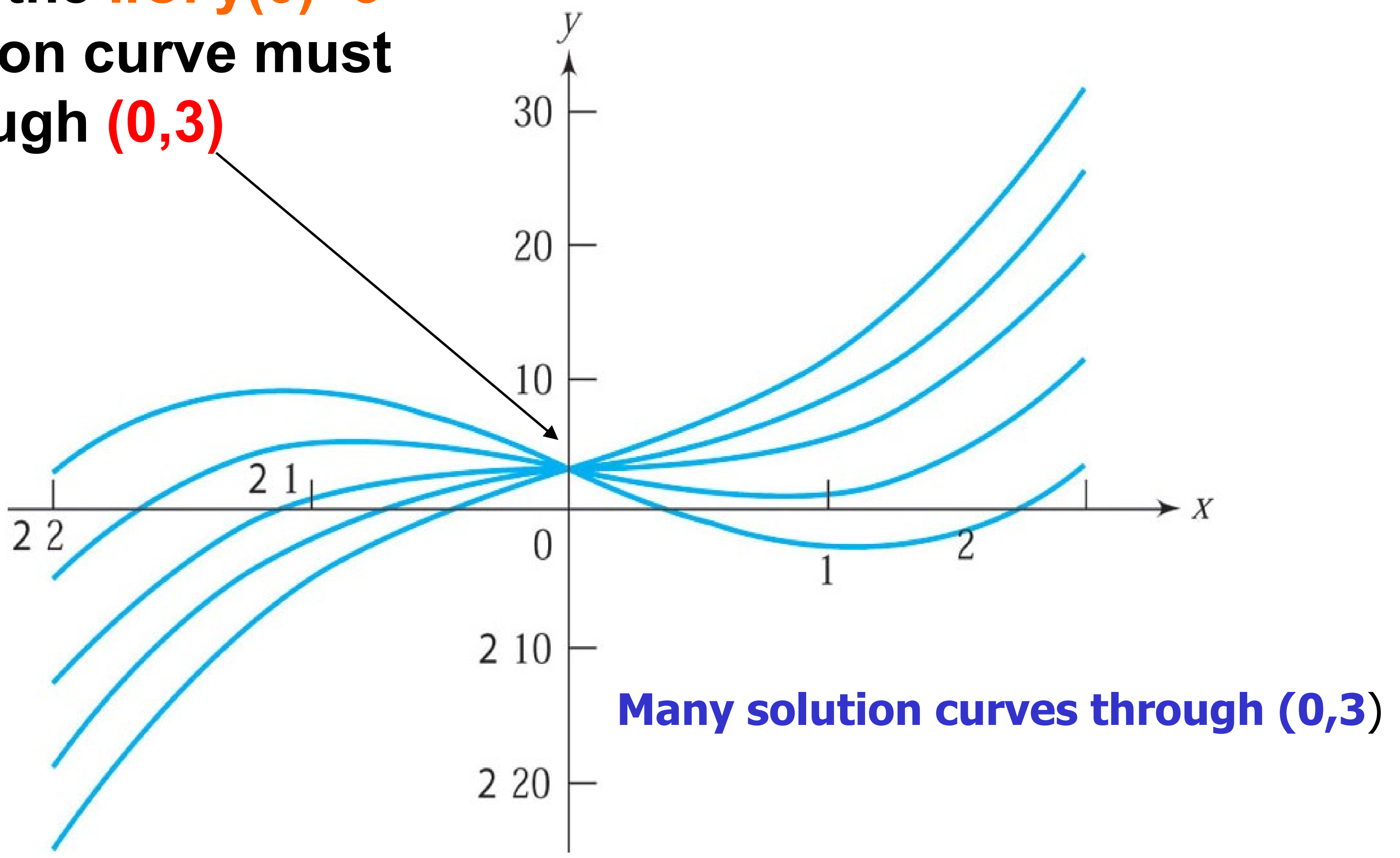


Figure 2.2 ***Graphs of $y = 2x^3 + Cx + 3$ for various values of C.***

To satisfy the I.C. $y(0)=3$, $y'(0)=-1$, the solution curve must pass through (0,3) having slope -1

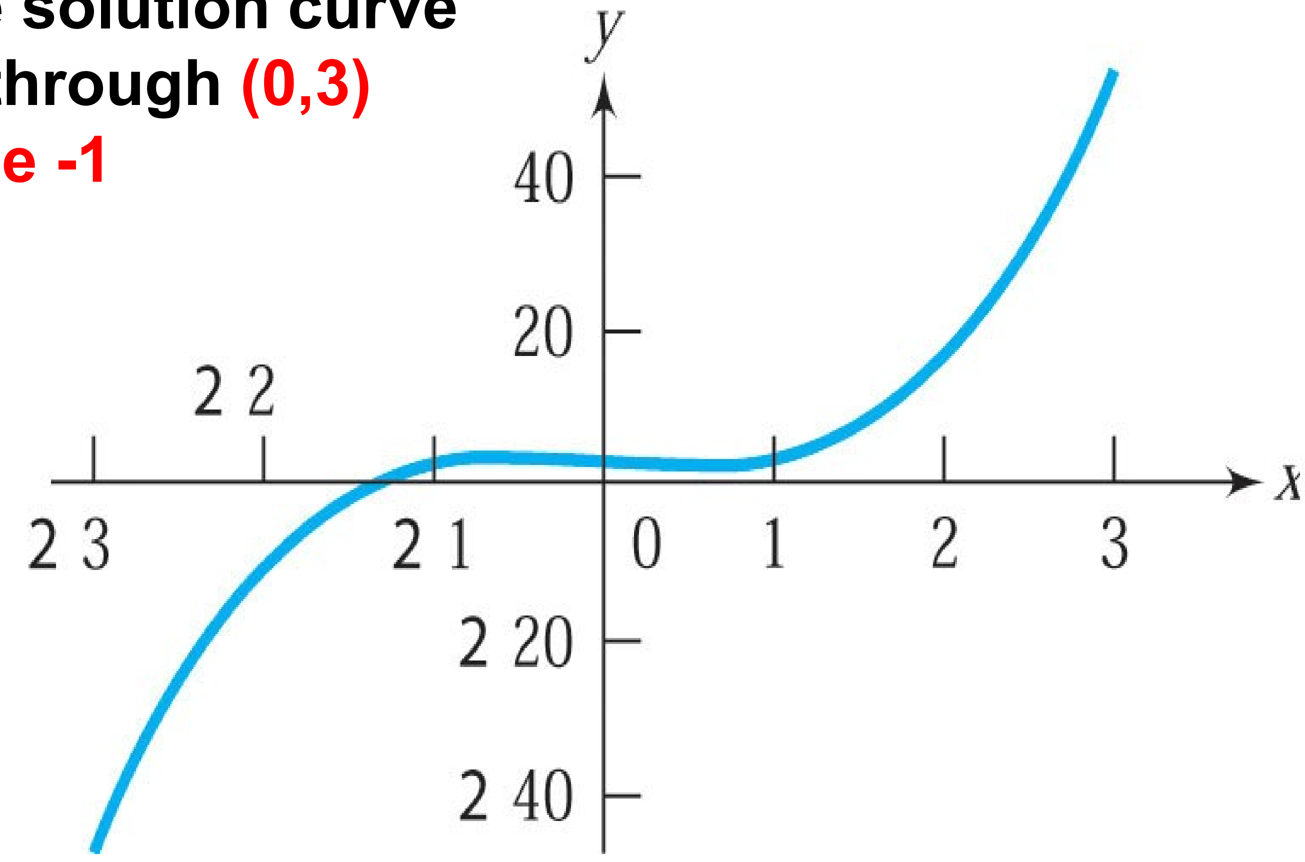


Figure 2.3 ***Graph of $y = 2x^3 - x + 3$.***

1.1 Definitions and Terminology

Solutions

General Solution: Solutions obtained from integrating the differential equations are called general solutions. The general solution of a nth order ordinary differential equation contains n arbitrary constants resulting from integrating   times.

Particular Solution: Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.

Singular Solutions: Solutions that can not be expressed by the general solutions are called singular solutions.

1.1 Definitions and Terminology

DEFINITION: **implicit solution**

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the **differential equation** on I.

➔a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

In contrast to an explicit solution $y = \phi(x)$

1.1 Definitions and Terminology

DEFINITION: **implicit solution**

Verify by implicit differentiation that the given equation implicitly defines a solution of the differential equation

$$y^2 + xy - 2x^2 - 3x - 2y = C$$

$$y - 4x - 3 + (x + 2y - 2)y' = 0$$

1.1 Definitions and Terminology

DEFINITION: **implicit solution**

Verify by implicit differentiation that the given equation implicitly defines a solution of the differential equation $y^2 + xy - 2x^2 - 3x - 2y = C$

$$y - 4x - 3 + (x + 2y - 2)y' = 0$$

$$d(y^2 + xy - 2x^2 - 3x - 2y)/dx = d(C)/dx$$

$$==> 2yy' + y + xy' - 4x - 3 - 2y' = 0$$

$$==> y - 4x - 3 + xy' + 2yy' - 2y' = 0$$

$$==> y - 4x - 3 + (x + 2y - 2)y' = 0$$

1.1 Definitions and Terminology

Conditions

Initial Condition: Constrains that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems.

Boundary Condition: Constrains that are specified at the boundary points, generally space points, are called boundary conditions. Problems with specified boundary conditions are called boundary value problems.

1.2 Initial-Value Problem

First- and Second-Order IVPS

Solve: $$\frac{dy}{dx} = f(x, y)$$

Subject to: $$y(x_0) = y_0$$

Solve: $$\frac{d^2y}{dx^2} = f(x, y, y')$$

Subject to: $$y(x_0) = y_0, \quad y'(x_0) = y_1$$

1.2 Initial-Value Problem

DEFINITION: **initial value problem**

An **initial value problem** or IVP is a problem which consists of an *n*-th order ordinary differential equation along with *n* initial conditions defined at a point x_0 found in the interval of definition I

differential equation $\dfrac{d^n y}{dx^n} = f(x,\ y,\ y',\ldots,y^{(n-1)})$

initial conditions $y(x_0) = y_0,\ \ y'(x_0) = y_1,\ldots,y^{(n-1)}(x_0) = y_{n-1}$

where $y_0,\ \ y_1,\ldots,y_{n-1}$ are known constants.

1.2 Initial-Value Problem

THEOREM: Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f / \partial y$ are continuous on R, Then there exists some interval $I_0 : x_0 - h < x < x_0 + h,\ h > 0$ contained in $a \le x \le b$ and a unique function $y(x)$ defined on I_0 that is a solution of the initial value problem.