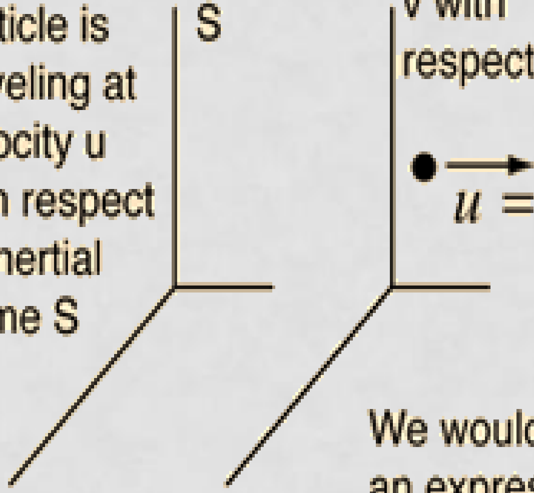
Addition of velocities

In relativistic physics, a **velocity-addition formula** is an equation that relates the velocities of objects in different reference frames. It is 3-dimensional in nature. It also relates different frames, that is, the formulas applies to successive Lorentz transformation. Accompanying velocity addition is a kinematic effect called the Thomas precession. Successive non-collinear Lorentz boosts affects a rotation to the coordinate system

No two objects can have a relative velocity greater than c! But what if I observe a spacecraft traveling at 0.8c and it fires a projectile which it observes to be moving at 0.7c with respect to it!? Velocities must transform according to the Lorentz transformation, and that leads to a very non-intuitive result called Einstein velocity addition.

Assume a particle is traveling at velocity u with respect to inertial frame S



S' moving at velocity v with respect to S

$$u = \frac{dx}{dt}$$

Lorentz transformation

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We would like an expression for $u' = \frac{dx'}{dt'}$ which is the velocity as measured in the moving reference frame S'

Just taking the differentials of these quantities leads to the velocity transformation. Taking the differentials of the Lorentz transformation expressions for x' and t' above gives

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{vdx}{c^2})} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\frac{dx}{dt}}{c^2}}$$

Putting this in the notation introduced in the illustration above:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

The reverse transformation is obtained by just solving for u in the above expression. Doing that gives

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

The formulas for boosts in the standard configuration follow most straight forwardly from taking differentials of the inverse Lorentz boost in standard configuration,

$$dx = \gamma(dx' + vdt'), \quad dy = dy', \quad dz = dz', \quad dt = \gamma\left(dt' + \frac{V}{c^2}dx'\right).$$

Divide the first three equations by the fourth,

$$\frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + \frac{V}{c^2}dx')}, \quad \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{V}{c^2}dx')}, \quad \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + \frac{V}{c^2}dx')},$$

or

$$\frac{dx}{dt} = \frac{dx' + vdt'}{dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'}')}, \quad \frac{dy}{dt} = \frac{dy'}{\gamma dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'})}, \quad \frac{dz}{dt} = \frac{dz'}{\gamma dt'(1 + \frac{V}{c^2}\frac{dx'}{dt'})},$$

which is

$$v_x = \frac{v'_x + V}{1 + \frac{V}{c^2} v'_x}, \quad v_y = \frac{\sqrt{1 - \frac{V^2}{c^2}} v'_y}{1 + \frac{V}{c^2} v'_x}, \quad v_z = \frac{\sqrt{1 - \frac{V^2}{c^2}} v'_z}{1 + \frac{V}{c^2} v'_x}$$