

Mass energy equivalence.

By the Work-Energy Theorem:

$$W = KE$$

Where W is work done, KE is kinetic energy,

In this case, we will consider $PE = 0$

Where PE is potential energy

so:

$$W = KE = E$$

From the definition of work:

$$W = F(Dx)$$

$$E = F(Dx)$$

Therefore:

$$E = \int_0^x F dx$$

From Newton's original statement of $F=ma$:

$$F = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$F = \frac{d}{dt}(mv)$$

This, along with the previous equation yields:

$$E = \int_0^x F dx$$

We have dx and dt in the equation.
Let's write it all in terms of dt :

$$v = \frac{Dx}{Dt} = \frac{dx}{dt}$$

$$dx = v dt$$

When the variable changes from dx to dt ,
we must also change the bounds of the integral

$$E = \int_0^x \frac{d}{dt} (mv) dx = \int_0^t \frac{d}{dt} (mv) v dt$$

We can eliminate the dt 's so:

$$E = \int_0^t \frac{d}{dt} (mv) v dt = \int_0^{mv} v d(mv)$$

Notice that the dt changed to $d(mv)$ so the bounds also changed

For velocities approaching c , the mass increases

The relativistic mass is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E = \int_0^{mv} v d(mv) = \int_0^{mv} v d\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Since m_0 is a constant, it can be factored out of the integral:

$$E = \int_0^{mv} v d\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \int_0^v v d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Applying the Quotient Rule:

(keep in mind v is a variable and c is a constant)

$$d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{(dv)\sqrt{1 - \frac{v^2}{c^2}} - v\left(\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right)(2v dv)}{\left(1 - \frac{v^2}{c^2}\right)}$$

Combining these expressions for Energy yields:

$$E = m_0 \int_0^v v d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

$$E = m_0 \int_0^v \frac{(dv) \sqrt{1 - \frac{v^2}{c^2}} - v \left(\frac{1}{2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left(-\frac{1}{c^2}\right) (2v dv)}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \right) dv$$

To get a common denominator, we will multiply the numerator

and denominator of

$$\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{by} \quad \frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{v^2}{c^2})}$$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \right) dv$$

$$E = m_0 \int_0^v \left[\frac{(1 - \frac{v^2}{c^2})v}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \right] dv$$

Simplifying:

$$E = m_0 \int_0^v \left[\frac{\left(1 - \frac{v^2}{c^2}\right)v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \right] dv$$

$$E = m_0 \int_0^v \frac{v - \frac{v^3}{c^2} + \frac{v^3}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\left(\frac{c^2}{c^2} - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\left(\frac{c^2 - v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\frac{(c^2 - v^2)^{\frac{3}{2}}}{c^3}} dv$$

$$E = m_0 \int_0^v \frac{v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

$$c^3$$

$$E = m_0 \int_0^v \frac{c^3 v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

$$E = m_0 c^3 \int_0^v \frac{v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

To evaluate the integral we make the substitution:

$$u = c^2 - v^2$$

$$\frac{du}{dv} = -2v$$

$$-\frac{1}{2} du = v dv$$

Therefore:

$$E = m_0 c^3 \int_0^v \frac{v dv}{(c^2 - v^2)^{\frac{3}{2}}}$$

$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} \frac{-\frac{1}{2} du}{u^{\frac{3}{2}}}$$

$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} - \frac{1}{2} \frac{du}{u^{\frac{3}{2}}}$$

$$E = - \frac{m_0 c^3}{2} \int_{c^2}^{c^2 - v^2} \frac{du}{u^{\frac{3}{2}}}$$

$$E = - \frac{m_0 c^3}{2} \left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_{c^2}^{c^2 - v^2} \right)$$

$$E = m_0 c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

$$E = m_0 c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

$$E = m_0 c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{c} \right)$$

$$E = m_0 c^3 \left(\frac{c}{c\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_0 c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_0 c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c \sqrt{c^2 - v^2}} \right)$$

$$E = m_0 c^2 \left(\frac{c - \sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}} \right)$$

$$E = m_0 c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}} \right)$$

$$E = m_0 c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\frac{1}{c} \sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{\left(\frac{1}{c^2}\right)(c^2 - v^2)}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} - 1 \right)$$

$$E = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

$$E = c^2 \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 \right)$$

Recall the relativistic mass:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting this yields:

$$E = c^2 (m - m_0)$$

or

$$E = (m - m_0)c^2$$

$$E = mc^2$$