UNIT-II

2.1 Modular Arithmetic

Modular arithmetic is 'clock arithmetic' a **congruence** a = b mod n says when divided by *n* that a and b have the same remainder

100 = 34 mod 11 usually

have 0<=b<=n-1

-12mod7 = -5mod7 = 2mod7 = 9mod7

b is called the **residue** of a mod n

can do arithmetic with integers modulo n with all results between 0 and n

Addition

a+b mod n

Subtraction

a-b mod n = a+(-b) mod n

Multiplication

a.b mod n

derived from repeated addition

can get a.b=0 where neither a,b=0

- eg 2.5 mod 10

Division

a/b mod n

is multiplication by inverse of b: $a/b = a.b^{-1} \bmod n$

if n is prime $b^{-1} \bmod n$ exists s.t $b.b^{-1} = 1 \bmod n$

- eg 2.3=1 mod 5 hence 4/2=4.3=2 mod 5

· integers modulo n with addition and multiplication form a commutative ring with the laws of

Associativity

(a+b)+c = a+(b+c) mod n

Commutativity

a+b = b+a mod n

Distributivity

$(a+b).c = (a.c)+(b.c) \bmod n$

· also can chose whether to do an operation and then reduce modulo n, or reduce then do the operation, since reduction is a homomorphism from the ring of integers to the ring of integers modulo n

- $a+/-b \bmod n = [a \bmod n +/- b \bmod n] \bmod n$
- (the above laws also hold for multiplication)

· if n is constrained to be a prime number *p* then this forms a **Galois Field modulo p** denoted **GF(p)** and all the normal laws associated with integer arithmetic work

2.1.1 Exponentiation in GF(p)

· many encryption algorithms use exponentiation - raising a number a (base) to some power b (exponent) mod p

- $b = a^e \bmod p$

exponentiation is basically repeated multiplication, which take s O(n) multiples for a number n

a better method is the square and multiply algorithm

let base = a, result =1
for each bit ei (LSB to MSB) of exponent
if ei=0 then
square base mod
p if ei=1 then
multiply result by base mod p
square base mod p (except for
MSB) required ae is result

· only takes $O(\log_2 n)$ multiples for a number n

see Sebbery p9 Fig2.1 + example

2.1.2 Discrete Logarithms in GF(p)

· the inverse problem to exponentiation is that of finding the **discrete logarithm** of a number modulo p

- find x where $a^x = b \bmod p$

Seberry examples p10

· whilst exponentiation is relatively easy, finding discrete logarithms is generally a **hard** problem, with no easy way

· in this problem, we can show that if p is prime, then there always exists an a such that there is always a discrete logarithm for any b!=0

- successive powers of a "generate" the group mod p

such an a is called a **primitive root** and these are also relatively hard to find

2.1.3 Greatest Common Divisor

the greatest common divisor (a,b) of a and b is the largest number that divides evenly into both a and b

Euclid's Algorithm is used to find the Greatest Common Divisor (GCD) of two numbers a and n, a<n

- use fact if a and b have divisor d so does a-b, a-2b

GCD (a,n) is given by:
let g0=n
g1=a
gi+1 = gi-1 mod gi
when gi=0 then (a,n) = gi-1
eg find (56,98)

g0=98
g1=56
g2 = 98 mod 56 = 42
g3 = 56 mod 42 = 14
g4 = 42 mod 14 = 0
hence (56,98)=14

2.1.4 Inverses and Euclid's Extended GCD Routine

· unlike normal integer arithmetic, sometimes a number in modular arithmetic has a unique inverse

- a^{-1} is inverse of a mod n if $a.a^{-1} = 1 \bmod n$
- where a,x in {0,n-1}
- eg 3.7 = 1 mod 10

if (a,n)=1 then the inverse always exists

can extend **Euclid's Algorithm** to find Inverse by keeping track of $g_i = u_i.n + v_i.a$

Extended Euclid's (or Binary GCD) **Algorithm** to find Inverse of a number a mod n (where (a,n)=1) is:

Inverse(a,n) is given
by: g0=n u0=1 v0=0
g1=a u1=0 v1=1

let
y = gi-1 div gi
gi+1 = gi-1 - y.gi = gi-1 mod
gi ui+1 = ui-1 - y.ui
vi+1 = vi-1 - y.vi
when gi=0 then Inverse(a,n) = vi-1

Example

eg: want to find Inverse(3,460):

i	y	g	u	v
0	-	460	1	0
1	-	3	0	1
2	153	1	1	-153
3	3	0	-3	460

hence Inverse(3,460) = -153 = 307 mod 460

2.1.5 Euler Totient Function [[phi]](n)

· if consider arithmetic modulo n, then a **reduced set of residues** is a subset of the complete set of residues modulo n which are relatively prime to n

- eg for n=10,
- the complete set of residues is {0,1,2,3,4,5,6,7,8,9}
- the reduced set of residues is {1,3,7,9}

the number of elements in the reduced set of residues is called the **Euler Totient function [[phi]](n)**

there is no single formula for [[phi]](n) but for various cases count how many elements are excluded[4]:

p (p prime)	[[phi]](p) =p-1
pr (p prime)	[[phi]](p) =pr-1(p-1)
p.q (p,q prime)	[[phi]](p.q) =(p-1)(q-1)

see Seberry Table 2.1 p13

several important results based on [[phi]](n) are:

Theorem (Euler's Generalization)

- let gcd(a,n)=1 then
- $a^{[[phi]](n)} \bmod n = 1$

Fermat's Theorem

- let p be a prime and gcd(a,p)=1 then
- $a^{p-1} \bmod p = 1$

Algorithms to find **Inverses** $a^{-1} \bmod n$

search 1,...,n-1 until an a^{-1} is found with $a.a^{-1} \bmod n$

if [[phi]](n) is known, then from Euler's Generalization

§ $$a^{-1} = a^{[[phi]](n)-1} \bmod n$$

otherwise use Extended Euclid's algorithm for inverse

2.1.6 Computing with Polynomials in GF(qn)

have seen arithmetic modulo a prime number GF(p)

also can do arithmetic modulo q over polynomials of degree n, which also form a **Galois Field GF(q^n)**

its elements are polynomials of degree (n-1) or lower

- $a(x)=a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$

have residues for polynomials just as for integers

- $p(x)=q(x)d(x)+r(x)$
- and this is unique if deg[r(x)]<deg[d(x)]

if r(x)=0, then d(x) **divides** p(x), or is a **factor** of p(x)

addition in GF(q^n) just involves summing equivalent terms in the polynomial modulo q (XOR if q=2)

- $a(x)+b(x)=(a_{n-1}+b_{n-1})x^{n-1}+...+(a_1+b_1)x+(a_0+b_0)$

2.1.7 Multiplication with Polynomials in GF(qn)

multiplication in GF(q^n) involves [5]

- multiplying the two polynomials together (cf longhand multiplication; here use shifts & XORs if q=2)
- then finding the residue modulo a given **irreducible polynomial** of degree n

an **irreducible polynomial** d(x) is a 'prime' polynomial, it has no polynomial divisors other than itself and 1

modulo reduction of p(x) consists of finding some r(x) st: p(x)=q(x)d(x)+r(x)

- nb. in GF(2^n) with $d(x)=x^3+x+1$ can do simply by replacing x^3 with x+1

eg in GF(2^3) there are 8 elements:

- 0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1

with irreducible polynomial d(x)=x^3+x+1* arithmetic in this field can be summarised as:

Seberry Table 2.3 p20

can adapt GCD, Inverse, and CRT algorithms for GF(q^n)

- [[phi]](p(x)) = 2^n-1 since every poly except 0 is relatively prime to p(x)

· arithmetic in GF(q^n) can be much faster than integer arithmetic, especially if the irreducible polynomial is carefully chosen

- eg a fast implementation of GF(2^{127}) exists

· has both advantages and disadvantages for cryptography, calculations are faster, as are methods for breaking

Public-Key Ciphers

traditional **secret key** cryptography uses a single key shared by both sender and receiver

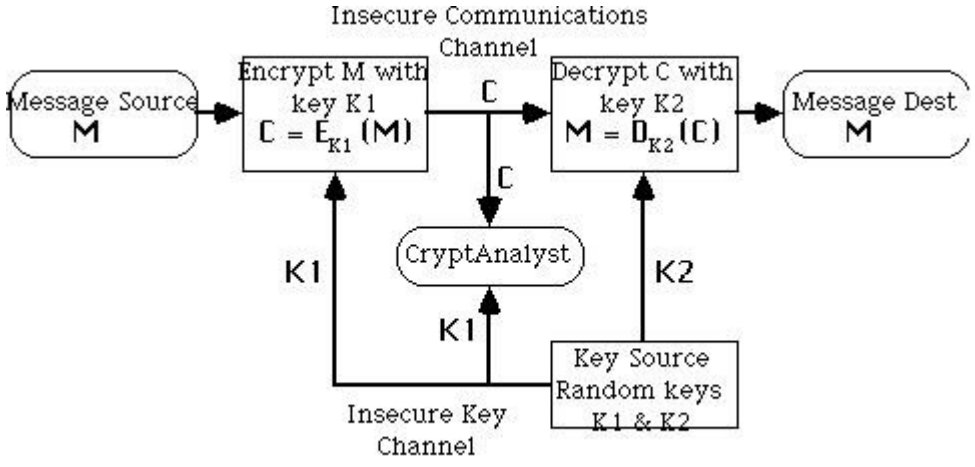
if this key is disclosed communications are compromised

also does not protect sender from receiver forging a message & claiming is sent by sender, parties are equal

public-key (or **two-key**) **cryptography** involves the use of two keys:

- a **public-key**,whichmaybeknownbyanybody,andcanbeusedto**encrypt messages**, and **verify signatures**

- a **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**



Asymmetric (Public-Key) Encryption System

the public-key is easily computed from the private key and other information about the cipher (a polynomial time (P-time) problem)

however, knowing the public-key and public description of the cipher, it is still computationally infeasible to compute the private key (an NP-time problem)

thus the public-key may be distributed to anyone wishing to communicate securly with its owner (although secure distribution of the public-key is a non-trivial problem - the **key distribution** problem)

have three important classes of public-key algorithms:

- **Public-Key Distribution Schemes** (PKDS) - where the scheme is used to securely exchange a single piece of information (whose value depends on the two parties, but cannot be set).
- This value is normally used as a session key for a private-key scheme
- **Signature Schemes** - used to create a digital signature only, where the private-key signs (create) signatures, and the public-key verifies signatures
- **Public Key Schemes (PKS)** - used for encryption, where the public-key encrypts messages, and the private-key decrypts messages.
- Any public-key scheme can be used as a PKDS, just by selecting a message which is the required session key
- Many public-key schemes are also signature schemes (provided encryption& decryption can be done in either order)

2.2.1 RSA Public-Key Cryptosystem

· best known and widely regarded as most practical public-key scheme was proposed by Rivest, Shamir & Adleman in 1977:

R L Rivest, A Shamir, L Adleman, "On Digital Signatures and Public Key Cryptosystems", Communications of the ACM, vol 21 no 2, pp120-126, Feb 1978

it is a public-key scheme which may be used for encrypting messages, exchanging keys, and creating digital signatures

is based on exponentiation in a finite (Galois) field over integers modulo a prime

- nb exponentiation takes $O((\log n)^3)$ operations

its security relies on the difficulty of calculating factors of large numbers

- nb factorization takes $O(e^{\log n \log \log n})$ operations
- (same as for discrete logarithms)

· the algorithm is patented in North America (although algorithms cannot be patented elsewhere in the world)

- this is a source of legal difficulties in using the scheme

RSA is a public key encryption algorithm based on exponentiation using modular arithmetic

to use the scheme, first generate keys:

Key-Generation by each user consists of:

- selecting two large primes at random (~100 digit), p, q
- calculating the system modulus R=p.q p, q primes
- selecting at random the encryption key e,
- e < R, gcd(e, F(R)) = 1
- solving the congruence to find the decryption key d,
- e.d [[equivalence]] 1 mod [[phi]](R) 0 <= d <= R
- publishing the public encryption key: K1={e,R}
- securing the private decryption key: K2={d,p,q}

Encryption of a message M to obtain ciphertext C is:

$C = M^e \bmod R$ 0 <= d <= R

Decryption of a ciphertext C to recover the message M is:

- $M = C^d = M^{e.d} = M^{1+n.[[phi]](R)} = M \bmod R$

the RSA system is based on the following result:

if R = pq where p, q are distinct large primes then
X [[phi]](R) = 1 mod R for
all x not divisible by p or q
and [[Phi]](R) = (p-1)(q-1)

RSA Example

usually the encryption key e is a small number, which must be relatively prime to [[phi]](R) (ie GCD(e, [[phi]](R)) = 1)

typically e may be the same for all users (provided certain precautions are taken), 3 is suggested

the decryption key d is found by solving the congruence:

e.d [[equivalence]] 1 mod [[phi]](R), 0 <= d <= R,

an extended Euclid's GCD or Binary GCD calculation is done to do this.

given e=3, R=11*47=517, [[phi]](R)=10*46=460
then d=Inverse(3,460) by Euclid's alg:

i	y	g	u	v
0	-460			10
1	-	3	0	1

2 153 1 1 -153
3 3 0 -3460
ie: d = -153, or 307 mod 517

a sample RSA encryption/decryption calculation is:

M = 26
C = 263 mod 517 = 515 M
= 515307 mod 517 = 26

.

2.2.1.2 Security of RSA

The security of the RSA scheme rests on the difficulty of factoring the modulus of the scheme R

best known factorization algorithm (Brent-Pollard) takes:

$$O\left(\frac{e^{\sqrt{2 \ln p \ln \ln p}}}{\ln p}\right)$$

operations on number R whose largest prime factor is p

Decimal Digits in R	#Bit Operations to Factor R
20	7200
40	3.11e+06
60	4.63e+08
80	3.72e+10
100	1.97e+12
120	7.69e+13
140	2.35e+15
160	5.92e+16
180	1.26e+18
200	2.36e+19

· This leads to R having a length of 200 digits (or 600 bits) given that modern computers perform 1-100 MIPS the above can be divided by 10^6 to get a time in seconds

- nb: currently 1e+14 operations is regarded as a limit for computational feasability and there are 3e+13 usec/year

but most (all!!) computers can't directly handle numbers larger than 32-bits (64-bits on the very newest)

hence need to use **multiple precision arithmetic** libraries to handle numbers this large

2.2.1.3 Multi-Precision Arithmetic

involves libraries of functions that work on multiword (multiple precision) numbers

classic references are in Knuth vol 2 - "Seminumerical Algorithms"

- multiplication digit by digit
- do exponentiation using square and multiply[6]

are a number of well known multiple precision libraries available - so don't reinvent the wheel!!!!

can use special tricks when doing modulo arithmetic, especially with the modulo reductions

2.2.1.4 Faster Modulo Reduction

* Chivers (1984) noted a fast way of performing modulo reductions whilst doing multi-precision arithmetic calcs

Given an integer A of n characters (a_0, ... , a_{n-1}) of base b

$$A = \sum_{i=0}^{n-1} a_i b^i \text{ then}$$

$$A \equiv \left\{ \sum_{i=0}^{n-2} a_i b^i + a_{n-1} b^{n-1} \pmod{jm} \right\} \pmod{m}$$

ie: this implies that the MSD of a number can be removed and its remainder mod m added to the remaining digits will result in a number that is congruent mod m to the original.

* Chivers algorithm for reducing a number is thus:

Construct an array $\mathbf{R = (b^d, 2.b^d, \ldots , (b-1).b^d)(mod\ m)}$

FOR i = n-1 to d do

WHILE A[i] != 0 do

j = A[i];

A[i] = 0;

$A = A + b^{i-d}.R[j];$

END WHILE

END FOR

where A[i] is the i^{th} character of number A

R[j] is the j^{th} integer residue from the array R

n is the number of symbols in A

d is the number of symbols in the modulus

2.2.1.5 Speeding up RSA - Alternate Multiplication Techniques

conventional multiplication takes $O(n^2)$ bit operations, faster techniques include:

the Schonhage-Strassen Integer Multiplication Algorithm:

- breaks each integer into blocks, and uses them as coefficients of a polynomial
- evaluates these polynomials at suitable points, & multiplies the resultant values
- interpolates these values to form the coefficients of the product polynomial
- combines the coefficients to form the product of the original integer
- the Discrete Fourier Transform, and the Convolution Theorem are used to speed up the interpolation stage
- can multiply in O(n log n) bit operations

the use of specialized hardware because:

- conventional arithmetic units don't scale up, due to carry propogation delays
- so can use serial-parallel carry-save, or delayed carry-save techniques with O(n) gates to multiply in O(n) bit operations,
- or can use parallel-parallel techniques with $O(n^2)$ gates to multiply in O(log n) bit operations

2.2.1.6 RSA and the Chinese Remainder Theorem

· a significant improvement in decryption speed for RSA can be obtained by using the Chinese Remainder theorem to work modulo p and q respectively

- since p,q are only half the size of R=p.q and thus the arithmetic is much faster

CRT is used in RSA by creating two equations from the decryption calculation:

$$M = Cd \bmod R$$

as follows:

$$M1 = M \bmod p = (C \bmod p)d \bmod (p-1)$$
$$M2 = M \bmod q = (C \bmod q)d \bmod (q-1)$$

then the pair of equations

$$M = M1 \bmod p \quad M = M2 \bmod q \text{ has}$$

a unique solution by the CRT, given by:

$$M = [((M2 + q - M1)u \bmod q] p + M1$$

where

$$p.u \bmod q = 1$$

2.2.1.7 Primality Testing and RSA

The first stage of key-generation for RSA involves finding two large primes p, q

Because of the size of numbers used, must find primes by trial and error

Modern primality tests utilize properties of primes eg:

- $a^{n-1} = 1 \bmod n$ where GCD(a,n)=1
- all primes numbers 'n' will satisfy this equation
- somecompositenumberswillalsosatisfytheequation,andarecalledpseudo-primes.

Most modern tests guess at a prime number 'n', then take a large number (eg 100) of numbers 'a', and apply this test to each. If it fails the number is composite, otherwise it is is probably prime.

There are a number of stronger tests which will accept fewer composites as prime than the above test. eg:

$$\text{GCD}(a,n) = 1, \quad \text{and} \quad \left(\frac{a}{n}\right) (\text{mod } n) = a^{\frac{(n-1)}{2}} (\text{mod } n)$$

$$\text{where } \left(\frac{a}{n}\right) \text{ is the Jacobi symbol}$$

2.2.1.8 RSA Implementation in Practice

Software implementations

- generally perform at 1-10 bits/second on block sizes of 256-512 bits
- two main types of implementations:
  - § - on micros as part of a key exchange mechanism in a hybrid scheme
  - § - on larger machines as components of a secure mail system

Harware Implementations

- generally perform 100-10000 bits/sec on blocks sizes of 256-512 bits
- all known implementations are large bit length conventional ALU units

ElGamal

A variant of the Diffie-Hellman key distribution scheme, allowing secure exchange of messages

published in 1985 by ElGamal in

T. ElGamal, "A Public Key Cryptosystem and a Signature Scheme Based on Discrete Logarithms", IEEE Trans. Information Theory, vol IT-31(4), pp469-472, July 1985.

like Diffie-Hellman its security depends on the difficulty of factoring logarithms

Key Generation

- select a large prime p (~200 digit), and
- [[alpha]] a primitive element mod p
- A has a secret number x_A
- B has a secret number x_B
- A and B compute y_A and y_B respectively, which are then made public
  - $y_A = [[alpha]]^{xA} \bmod p$
  - $y_B = [[alpha]]^{xB} \bmod p$

to **encrypt** a message **M** into ciphertext **C**,

- selects a random number **k,** $0 <= k <= p-1$
- computes the message key **K**
  - $K = y_B^{k} \bmod p$
- computes the ciphertext pair: C = {c1,c2}
  - $C_1 = [[alpha]]^{k} \bmod p$ $C_2 = K.M \bmod p$

to **decrypt** the message

- extracts the message key **K**
  - $K = C_1^{xB} \bmod p = [[alpha]]^{k.xB} \bmod p$
- extracts **M** by solving for M in the following equation:
  - $C_2 = K.M \bmod p$

Other Public-Key Schemes

a number of other public-key schemes have been proposed, some of the better known being:

- Knapsack based schemes
- McEleice's Error Correcting Code based schems

ALL of these schemes have been broken

the only currently known secure public key schemes are those based on exponentiation (all of which are patented in North America)

it has proved to be very difficult to develop secure public key schemes

this in part is why they have not been adopted faster, as their theorectical advantages might have suggested